J. Fluid Mech. (1986), *vol.* 167, *pp.* 327–351
Printed in Great Britain

On the nonlinear critical layer below a nonlinear unsteady surface wave

By R. S. JOHNSON

School of Mathematics, The University, Newcastle upon Tyne, NE1 7RU, UK

(Received 10 April 1985)

A theory is presented for the development of the nonlinear critical layer below an unsteady free surface wave, of amplitude ϵ, described by the Korteweg–de Vries (KdV) equation. The problem is formulated (via the Euler equations) for wave propagation over an arbitrary shear in a two-dimensional channel which contains a critical level. The equations are scaled so as to be valid in the far-field regime of the surface wave, appropriate to the existence of the KdV equation, i.e. long waves. The regions above and below the critical layer are solved (to $O(\epsilon)$, as $\epsilon \to 0$) and thence expanded in the neighbourhood of the critical layer itself. The symmetry of the critical layer solution, assuming that it exists, is then sufficient to determine the Burns integral for the linearized wave speed, and the relevant KdV equation. These turn out to be the classical results evaluated in terms of finite parts.

The critical layer, of thickness $O(\epsilon^{\frac{1}{2}})$, is analysed to $O(\epsilon)$ and matched to the outer regions of the flow field. The initial configuration is taken to contain no closed streamlines, and so the vorticity can, presumably, be assigned from the undisturbed conditions at infinity. The initial surface profile must therefore contain a single peak, but by virtue of the KdV equation this can evolve into any number of solitons. Between consecutive pairs of peaks there will now appear regions of closed streamlines (cat's-eyes) with known vorticity. No recourse to a viscous argument is necessary to uniquely determine this vorticity. However, it is shown that the vorticity cannot be prescribed arbitrarily at all orders, initially: the long-wave assumption imposes a certain structure on the problem, and then the continuity of stream function and particle velocity fixes the vorticity. This agrees with the work of Varley & Blyth (1983) on the hydraulic equations. The vorticity inside the separating streamlines is obtained to $O(\epsilon)$, but it is shown that for unsteady motion this asymptotic expansion is not uniformly valid as the bounding streamlines are approached. An alternative method, which exploits Varley & Blythe's approach, is used to confirm the correctness of our results away from these boundaries, and to indicate that a non-uniformity is present near the separating streamlines. Thus the model requires the inclusion of a vortex sheet; for steady flow a jump in vorticity is sufficient. The removal of the discontinuity by allowing a distortion of the main flow outside the critical layer is briefly discussed.

Some results are presented for the formation of a single cat's-eye by using the exact 2-soliton solution of the KdV equation.

1. Introduction

The occurrence of a critical layer below a surface wave which moves over a shear flow (and in many other flow configurations) is a well-understood phenomenon. This layer is a region in the neighbourhood of the point (or rather, line) at which the wave speed is equal to the speed in the shear flow. The conditions that must pertain for the existence of a critical layer, and the methods for accommodating the non-uniformity there, are problems which have proved amenable to the analytical treatment. In particular, we can use either nonlinearity of viscous stresses – or both – within the layer to provide a well-defined structure. The size (i.e. thickness) of the layer is governed by the physical parameters, and the choice between a predominantly viscous or predominantly nonlinear critical layer will depend on the relative magnitudes of these parameters. For example, if the wave amplitude parameter (ϵ), although small, is greatly in excess of the inverse Reynolds number (R^{-1}) then the layer will be dominated by the nonlinear contribution (see Benney & Bergeron 1969; Davis 1969; Haberman 1972). The limiting procedure is therefore to let $R \rightarrow \infty$ *first*, whilst retaining the critical layer with a thickness $O(\epsilon^{\frac{1}{2}})$, as $\epsilon \rightarrow O^{+}$; a viscous-dominated layer has a thickness $O(R^{-\frac{1}{3}})$ (see, for example, Reid 1965). Of course, if a nonlinear critical layer theory implies any discontinuities in vorticity then it might be argued that thin viscous regions must be incorporated to smooth out the solution. In fact the authors Benney & Bergeron and Davis used this important refinement as the only means available for determining the vorticity in the region(s) of closed streamlines (the flow being steady). Another approach was adopted by Haberman (1972) which involves allowing a distortion to the main stream outside the critical layer; we shall mention this possibility later.

It is well known that, for steady, inviscid flows with closed streamlines, there is no way of uniquely determining the constant vorticity there (the Prandtl–Batchelor theorem). Such steady flows then require the adoption of a more complete theory for the viscous fluid, as first expounded by Batchelor (1956). Here we shall be concerned with the strictly inviscid theory, and examine with some care how far this will take us in obtaining a complete solution of the Euler equations (albeit at the expense of allowing discontinuities in vorticity). In order to uniquely determine the vorticity we shall develop a theory for unsteady motion, and so introduce the *initial* vorticity distribution. In particular, we could start with a flow for which the streamlines are not closed initially, and then allow closed regions to appear as the surface wave propagates. (We shall see, however, that we cannot allow the vorticity to be arbitrarily assigned *a priori*.) Naturally, any application of the results obtained from this type of analysis to a real (viscous) fluid must be on the understanding that the timescales involved are far less than those associated with any significant contribution from the viscous effects.

A particular class of inviscid problems, one of which describes the time development of a long eddy, has been discussed by Varley & Blythe (1983). These authors make use of model equations which are in some sense valid for long waves. These equations, the so-called 'hydraulic equations', represent the pressure solely by the local hydrostatic pressure distribution at each point below the surface. This problem then admits of exact solutions, and a careful treatment is given of the various types of streamline that arise in the presence of a critical layer. They show that even for steady flows the appropriate continuity conditions alone enable the vorticity to be uniquely determined for 'long' regions. In a sense, the present work can be regarded as an extension of this unsteady long-wave aspect of their analysis, although our presentation can only be asymptotic (as $\epsilon \rightarrow 0$). We shall use the complete Euler equations, the flow is to be unsteady, the surface wave dispersive, and the vorticity is specified to be consistent with the initial flow configuration. As found by Varley & Blythe (1983), this approach is particularly straightforward for long waves.

The choice of an unsteady flow which will admit the formation of at least one region of closed streamlines is, perhaps, not immediately obvious. However, this turns out to be quite simple if we take advantage of the exact n-soliton solution of the Korteweg–de Vries (or KdV) equation. To this end, we shall construct a critical-layer theory in the far-field long-wave regime defined by times $O(\epsilon^{-\frac{3}{2}})$ and characteristic variable $O(\epsilon^{-\frac{1}{2}})$, as $\epsilon \to 0$. This is the asymptotic limit which contains the KdV equation as the leading-order description of the surface wave. The surface wave will therefore be nonlinear, dispersive and above all unsteady (although steady solutions of the KdV equation do, of course, exist; the extension of Benney–Bergeron/Davis theory to steady, periodic, finite-amplitude waves is discussed by Moore & Saffman 1982, and for steady stratified flows over small obstacles by Margolis & Su 1978.) The formulation of the KdV equation for surface waves over arbitrary shears in the absence of a critical layer is given by Freeman & Johnson (1970), and for the solitary wave alone by Benjamin (1962).

The development of a theory which couples the nonlinear critical layer and the KdV equation is not new. Redekopp (1977) has shown that the KdV equation, or modified KdV equation, can arise in the study of solitary Rossby waves; the author also invokes the usual viscous secularity condition in order to produce a unique vorticity distribution. The techniques were extended by Maslowe & Redekopp (1980) to long waves in stratified flows, again resulting in the KdV equation. However, the possibility of a long-wave structure imposing a certain vorticity distribution (as found by Varley & Blythe 1983) does not seem to play any role in these results. In fact Brown & Stewartson (1979) have suggested that some doubt exists as to the validity of the scenario described by Redekopp (1977): on sufficiently long times the soliton solution may not survive. Of course, the same criticism can be levelled at our work. We must assume that the viscous critical layer is much thinner than the nonlinear one; that the viscous diffusion of vorticity is on a timescale much greater than that for the wave evolution; that the mean shear diffusion is also on a long timescale; that the wall layer remains thin and therefore does not extend out to the critical layer. All these assumptions are legitimate if $R^{-1} \ll \epsilon^{\frac{5}{2}}$ (see Redekopp 1977; Brown & Stewartson 1979; Maslowe & Redekopp 1980), but on timescales in excess of $O(\epsilon^{-\frac{3}{2}})$ the whole structure of the solution which we shall present may altogether disappear. It should also be noted that the use of a no-slip condition at the wall might well have a significant effect (see Haberman 1972).

A region of closed streamlines can be established with just a little care. For example, suppose that initially we have a wave profile which decays at infinity and contains a single peak. According to the (exact) 2-soliton solution of the KdV equation this profile will evolve into two solitons, the larger to the right of the smaller one and both propagating to the right. As soon as two maxima have appeared the critical layer between the two peaks will contain closed streamlines; initially these same streamlines are composed of two sets of open lines, to the left and to the right of the point below the maximum amplitude, bounded by the 'separating' streamlines. However, the closed streamline region now formed is not quite the classical picture, since its bounding streamline extends from $-\infty$, and returns there. (The Kelvin cats'-eyes pattern (Kelvin 1880) is usually represented as a periodic patern of closed regions bounded by separating streamlines which extend from $-\infty$ to $+\infty$.) Nevertheless, it is still possible to produce a 'classical' cat's-eye, although in a technically less satisfactory manner. To see this, consider now an initial profile with two unequal peaks, the larger far to the left of the smaller. Although a region of closed streamlines exists initially, these lines asymptote to a set of parallel lines as the distance between the solitons increases. In consequence we may reasonably designate the vorticity as that associated with the undisturbed shear flow. As the larger soliton approaches the smaller (both propagating to the right), a symmetric profile is produced at one instant in time. If the amplitudes of the two initial solitons are chosen to lie in a certain range then the symmetric profile has twin peaks. Since the maxima are equal the region of closed streamlines is bounded by streamlines which extend from $-\infty$ to $+\infty$. It is fairly easy to see that certain 3-soliton solutions of the KdV equation can allow three equal maxima (and therefore two cats'-eyes) at one instant. More generally, any number of maxima – and therefore any number of cats'-eyes – can be generated from a suitable initial profile with a single maximum, but these maxima are all unequal. The initial single-peaked profile which then evolves into a profile with two or more peaks, coupled with the restriction to long waves, constitute the novel features here. This will enable us to describe the formation of closed-streamline regions with known vorticity without recourse to any argument based on a viscous fluid. Since the open-streamline pattern will actually persist for some time, before generating the closed-streamline pattern, the problems encountered by Stewartson (1978) (when a sinusoidal Rossby wave was forced on a shear layer at $t = 0$) would not seem to be relevant here.

To summarize, we shall present a theory of the nonlinear critical layer which is applicable to the far field of the surface wave. This far-field region is the one which contains the KdV equation as the leading approximation to the description of the surface wave ($\epsilon \rightarrow 0$). The asymptotic expansions valid in the layers above and below the critical layer are themselves expanded to provide the appropriate matching conditions for the solution of the critical-layer problem. It turns out that merely supposing the existence of a critical-layer solution is sufficient to enable both the linearized wave speed and the KdV equation to be completely determined. (This makes use of a symmetry of the critical layer.) The details of the asymptotic expansion valid within the critical layer will be developed to $O(\epsilon)$. This will enable us to discuss in some detail how the time evolution of the flow manifests itself, particularly in the formation of cats'-eyes, and also the form taken by the vorticity (which itself varies on the long timescale). Our aim is to provide as complete a theory as the Euler equations will allow: we shall admit discontinuities in vorticity, arguing that a thin viscous layer could presumably be introduced to provide a continuous structure across the separating streamline. However, we shall also briefly indicate how the removal of any discontinuity in vorticity is possible, without invoking the action of viscosity, by a suitable distortion of the main stream (see Haberman 1972; also Brown & Stewartson 1978). It is beyond the scope of this present study to attempt a definitive answer as to whether our solution of the Euler equations is, in fact, a limit solution of the Navier–Stokes equations as $R \rightarrow \infty$.

2. Formulation

The flow under consideration is two-dimensional, bounded by a rigid horizontal surface below and a free surface (at constant pressure) above; the acceleration of gravity is constant. The governing equations for the inviscid, incompressible fluid are therefore

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla p + \boldsymbol{F}, \quad \nabla\cdot\boldsymbol{u} = 0; \quad \boldsymbol{F} \equiv (0, -g);$$

$$p = \text{const}, \quad w = \frac{\mathrm{D}h}{\mathrm{D}t} \quad \text{on } z = h(x,t);$$

$$w = 0 \quad \text{on } z = 0,$$

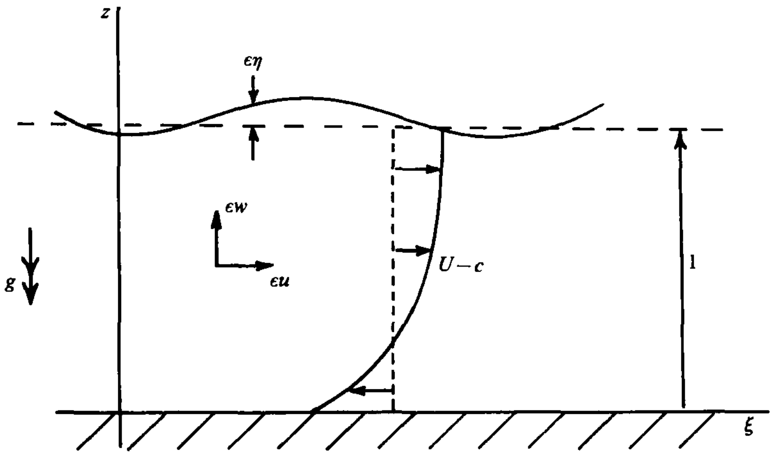

FIGURE 1. The non-dimensional variables.

written in the usual notation with $\boldsymbol{x} \equiv (x, z)$ and $\boldsymbol{u} \equiv (u, w)$. This system is non-dimensionalized by making use of the undisturbed depth of the fluid (h_0), a typical wavelength of the motion (λ), a typical wave amplitude (a) and a measure (the maximum value, say) of the shear profile beneath the surface. Such a scheme will generate two parameters: $\epsilon = a/h_0$, the amplitude parameter, and $\delta = h_0/\lambda$, the long-wave parameter. The non-dimensional variables are now scaled according to

$$\left.\begin{aligned} \xi = \left(\frac{\epsilon^{\frac{1}{2}}}{\delta}\right)(x-ct), \quad \tau = \left(\frac{\epsilon^{\frac{3}{2}}}{\delta}\right)t, \\ \text{with} \qquad u \to U(z)+\epsilon u, \quad w \to \epsilon w, \quad p \to P_{\mathrm{h}}(z)+\epsilon p, \\ h = 1+\epsilon\eta, \end{aligned}\right\} \tag{1}$$

where $P_{\mathrm{h}}(z)$ is the hydrostatic pressure distribution for $h = 1$. The limit process is then $\epsilon \to 0^+$, for arbitrary δ (and so ξ and τ are essentially defined using the scale-length h_0, rather than the original λ). We can note that the special case $\delta^2 = O(\epsilon)$ (often quoted in the derivation of the KdV equation) leaves $\xi = O(1)$ and $t = O(\epsilon^{-1})$. The transformations given in (1) require the (non-dimensional) shear profile, $U(z)$, to be specified; c is the speed of propagation of linear waves on the surface and is to be determined. For reference the non-dimensional scaled variables are used in figure 1, and the corresponding equations can be written as

$$\left.\begin{aligned} (U-c)\,u_\xi + U'w + \epsilon(u_\tau + uu_\xi + wu_z) + p_\xi = 0, \\ p_z + \epsilon(U-c)\,w_\xi + \epsilon^2(w_\tau + uw_\xi + ww_z) = 0, \\ u_\xi + w_z = 0; \\ \text{with} \qquad w = 0 \quad \text{on } z = 0 \\ \text{and} \qquad p = \eta, \quad w = (U-c)\,\eta_\xi + \epsilon(\eta_\tau + u\eta_\xi) \quad \text{on } z = 1+\epsilon\eta. \end{aligned}\right\} \tag{2}$$

The subscripts denote partial derivatives and $U' \equiv \mathrm{d}U/\mathrm{d}z$.

The asymptotic solution (as $\epsilon \to 0$) of (2) requires that each term in the expansions for u, w and p is defined for $0 \leqslant z \leqslant 1$. A critical layer then arises if $U(z) - c = 0$ for some $z \in (0, 1)$: we do not include the degenerate problem for which $U(z) - c = 0$ at $z = 0$ or 1. The existence of a critical layer means that, for example, the expansion for the stream function is not defined (at all orders) as $z \to z_c$, where $U(z_c) = c$. If, however, there is *no* critical layer, then a straightforward expansion solution

$$\boldsymbol{u} \sim \boldsymbol{u}_0 + \epsilon \boldsymbol{u}_1, \quad \eta \sim \eta_0 + \epsilon \eta_1, \quad p \sim p_0 + \epsilon p_1, \quad \text{as } \epsilon \to 0,$$

yields the KdV equation for $\eta_0(\xi, \tau)$:

$$-2I_3 \eta_{0\tau} + 3I_4 \eta_0 \eta_{0\xi} + J\eta_{0\xi\xi\xi} = 0, \tag{3}$$

where $$I_n = \int_0^1 [F(z)]^{-n} \, dz, \quad F(z) = U(z) - c,$$

and $$J = \int_0^1 \int_z^1 \int_0^{z_1} \frac{F^2(z_1)}{F^2(z)\, F^2(z_2)} \, dz_2 \, dz_1 \, dz,$$

(see Freeman & Johnson 1970). The linearized propagation speed c is determined by the integral constraint (Burns 1953)

$$I_2 = \int_0^1 [F(z)]^{-2} \, dz = 1. \tag{4}$$

In the present work we shall examine the solution of (2) when a critical layer is present, and for an initial surface profile which contains a single maximum value (although other choices are possible, as we mentioned earlier). The critical layer is assumed to exist, and to be at $z = z_c$ $(0 < z_c < 1)$, such that

$$F(z_c) = U(z_c) - c = 0, \quad F'(z_c) \neq 0.$$

Evaluation at the critical level will be denoted by the subscript c, e.g. $F_c = 0$, $F'_c \neq 0$. The equations (2) will, however, only describe the flow regime away from the critical layer (that is, above and below the layer), as $\epsilon \to 0^+$. The region containing the critical layer is represented by a suitably scaled version of (2). This scaling is well known and easily derived. Consider the total stream function ψ_T, for the flow, which can be written as

$$\psi_T \sim \int_{z_c}^{z} [U(z) - c] \, dz + \epsilon \psi_0 \quad \text{as } \epsilon \to 0^+,$$

where ψ_0 denotes the first approximation to the stream function as defined by equations (2). Now as $z \to z_c$ this yields

$$\psi_T \sim \tfrac{1}{2}(z - z_c)^2 F'_c + \epsilon \psi_0,$$

and, if ψ_0 remains bounded (and non-zero) as this limit is taken, then the asymptotic expansion is certainly non-uniform when $z - z_c = O(\epsilon^{\frac{1}{2}})$. (The behaviour of ψ_0 will be described later.) Thus we define new variables

$$z - z_c = \epsilon^{\frac{1}{2}} Z, \quad u = \epsilon^{-\frac{1}{2}} V, \quad w = W, \quad p = P, \tag{5}$$

to describe the nonlinear critical layer (see Benny & Bergeron 1969). The critical layer (or inner) region is therefore an appropriate solution of the set

$$\left.\begin{aligned} (\hat{U} - c) V_\xi + \epsilon^{\frac{1}{2}} (\hat{U}' W + V V_\xi + W V_Z + P_\xi) + \epsilon V_\tau &= 0, \\ P_Z + \epsilon^{\frac{1}{2}} (\hat{U} - c) W_\xi + \epsilon^2 (V W_\xi + W W_Z) + \epsilon^{\frac{5}{2}} W_\tau &= 0, \\ V_\xi + W_Z &= 0, \end{aligned}\right\} \tag{6}$$

where $\hat{U} = U(z_c + \epsilon^{\frac{1}{2}} Z)$, with $\epsilon \to 0^+$. We note that the original boundary conditions applicable to (2) are to be replaced here by conditions that allow matching to the outer regions.

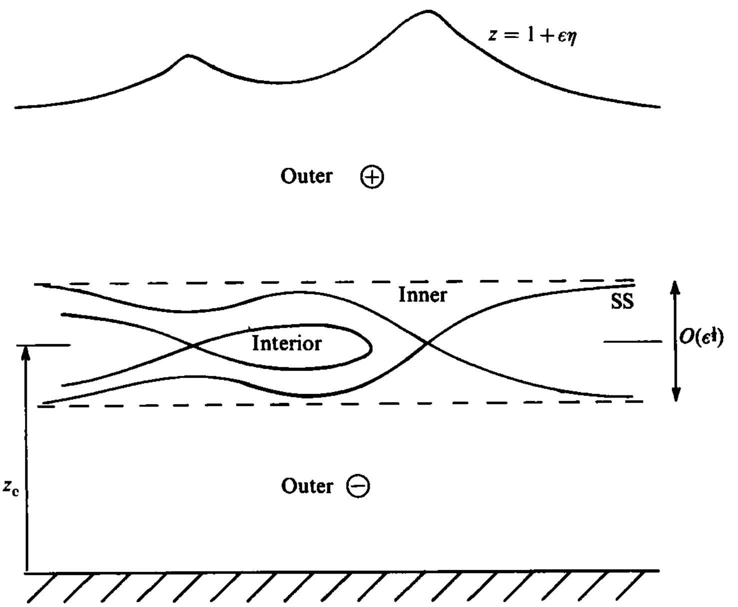

FIGURE 2. The asymptotic regions: outer above/below $(+/-)$; critical layer and separating streamlines (SS).

The procedure we adopt is first to solve (2), as $\epsilon \rightarrow 0^+$, in the regions above and below the critical layer; these outer regions will be denoted by superscripts $+$ and $-$ respectively. Of course, such solutions cannot be uniquely determined since they are, as yet, unconnected across the critical layer. However, these solutions can be expanded as $z \rightarrow z_c$ to yield the form that must be taken by the asymptotic expansion valid in the critical layer, and also to provide the matching conditions to be appended to (6). The critical-layer problem can now be analysed but, since we will allow a discontinuity in vorticity across a separating streamline, each term in the expansion takes (in general) one of two forms. This solution will, nevertheless, satisfy the conditions of continuity of both the stream function and the particle velocity across the separating streamline. The complete problem – both inner and outer – comprises five regions, which are depicted in figure 2, although the two regions inside the critical layer but outside the separating streamlines are identical by symmetry. Consequently the number of regions to be analysed is reduced to four, and of those the outer $\pm$ are rather similar. We shall comment, however, that the appropriate KdV equation and Burns condition can be derived without solving the critical-layer problem at all, provided only that the symmetric solutions do exist.

3. The outer regions

First we turn our attention to the solution of (2), in the regions above $(1 \geqslant z > z_c)$ and below $(0 \leqslant z < z_c)$ the critical layer. It is convenient to introduce the stream function $\psi(\xi, \tau, z; \epsilon)$ and then seek an asymptotic solution

$$\psi^{\pm} \sim \psi_0^{\pm} + \epsilon\psi_1^{\pm}; \quad p^{\pm} \sim p_0^{\pm} + \epsilon p_1^{\pm}; \quad \eta \sim \eta_0 + \epsilon\eta_1, \tag{7}$$

valid as $\epsilon \rightarrow 0^+$. The existence of an inner expansion, and its matching, as presented here does not necessitate terms in addition to the sequence $\{\epsilon^n\}$, with the possible exceptions of: (i) contributions to the arbitrary 'constant' (which may be a function of τ), and (ii) inclusion of an $O(\epsilon^{\frac{1}{2}})$ distortion of the main flow (see §5). Rather, the expansion of (7) as $z \to z_c$ completely determines the form of the inner expansion. The leading-order problem obtained by using (7) in (2) is described by

$$\left.\begin{aligned} &F\psi^{\pm}_{0\xi z} - F'\psi^{\pm}_{0\xi} + p^{\pm}_{0\xi} = 0; \quad p^{\pm}_{0z} = 0;\\ \text{with}\quad &p^{+}_{0} = \eta_0, \quad \psi^{+}_{0\xi} = -F_1\eta_{0\xi} \quad \text{on } z = 1,\\ \text{and}\quad &\psi^{-}_{0\xi} = 0 \text{ (or } \psi^{-}_{0} = 0\text{, say)} \quad \text{on } z = 0. \end{aligned}\right\} \tag{8}$$

Here, $F = F(z)$ and $F_1 = F(1)$; the solution of this set yields

$$\left.\begin{aligned} &p^{+}_{0} = \eta_0(\xi,\tau); \quad p^{-}_{0} = H_0(\xi,\tau);\\ &\psi^{+}_{0} = -\eta_0 F\left[1 - \int_z^1 F^{-2}\,\mathrm{d}z\right] + C^{+}_{0}; \quad \psi^{-}_{0} = -H_0 F\int_0^z F^{-2}\,\mathrm{d}z + C^{-}_{0}, \end{aligned}\right\} \tag{9}$$

where H_0 is an arbitrary function and $C^{\pm}_{0}(\tau)$ play the role of arbitrary 'constants'. Clearly, we could choose $C^{-}_{0} = 0$, for example, so that $z = 0$ is the streamline $\psi^{-}_{0} = 0$ (as mentioned in (8)). However, it is slightly neater to assign the constant associated with the critical-layer solution (as zero, say) and then evaluate $C^{\pm}_{n}$ accordingly. To express the above solution in a compact form, it is convenient to write η_0 as η^{+}_{0} and then set $H_0 = \eta^{-}_{0}$ (but still arbitrary, of course). If we then use the definitions

$$I^{+} = 1 - \int_z^1 F^{-2}\,\mathrm{d}z, \quad I^{-} = \int_0^z F^{-2}\,\mathrm{d}z, \tag{10}$$

the solution (9) becomes

$$p^{\pm}_{0} = \eta^{\pm}_{0}; \quad \psi^{\pm}_{0} = -F\eta^{\pm}_{0} I^{\pm} + C^{\pm}_{0}. \tag{11}$$

The problem defining the $O(\epsilon)$ terms is now given by

$$\left.\begin{aligned} &F\psi^{\pm}_{1\xi z} - F'\psi^{\pm}_{1\xi} + \psi^{\pm}_{0\xi\tau} + \psi^{\pm}_{0z}\psi^{\pm}_{0\xi z} - \psi^{\pm}_{0\xi}\psi^{\pm}_{0zz} + p^{\pm}_{1\xi} = 0,\\ &p^{\pm}_{1z} - F\psi^{\pm}_{0\xi\xi} = 0;\\ \text{with}\quad &p^{+}_{1} = \eta_1 \quad\text{and}\quad \psi^{+}_{1\xi} + \eta_0\psi^{+}_{0\xi z} = -(F\eta_{1\xi} + \eta_{0\tau} + F'\eta_0\eta_{0\xi} + \psi^{+}_{0z}\eta_{0\xi}) \quad \text{on } z = 1,\\ \text{and}\quad &\psi^{-}_{1\xi} = 0 \quad \text{on } z = 0. \end{aligned}\right\} \tag{12}$$

We have adopted the same notation as employed in (8), and the surface boundary conditions have been expanded about $z = 1$. If we write $\eta_1 = \eta^{+}_{1}$, and let $\eta^{-}_{1}(\xi,\tau)$ be an arbitrary function, then the compact form of the solution is

$$\left.\begin{aligned} p^{\pm}_{1} &= \eta^{\pm}_{1} \pm \eta^{\pm}_{0\xi\xi}\int^{\pm} F^2 I^{\pm}\,\mathrm{d}z;\\ \psi^{\pm}_{1} &= -FI^{\pm}\eta^{\pm}_{1} + F\eta^{\pm}_{0\xi\xi}\int^{\pm}\left(\int^{\pm} F^2 I^{\pm}\,\mathrm{d}z\right)\mathrm{d}z + \alpha^{\pm}_{1} + C^{\pm}_{1}\\ &\quad \pm F\int^{\pm} F^{-2}\left[(F^{-1} + F'I^{\pm})\int_\xi^\infty \eta^{\pm}_{0\tau}\,\mathrm{d}\xi + \tfrac{1}{2}(\eta^{\pm}_{0})^2\{(F^{-1} + F'I^{\pm})^2 - FF''(I^{\pm})^2\}\right]\mathrm{d}z \end{aligned}\right\} \tag{13}$$

where $\alpha^{-}_{1} = 0, \quad \alpha^{+}_{1} = \dfrac{F}{F_1}\left[\displaystyle\int_\xi^\infty \eta^{+}_{0\tau}\,\mathrm{d}\xi + \tfrac{1}{2}(\eta^{+}_{0})^2(F'_1 + 2F_1^{-1})\right],$

and

$$\int^{+} \mathrm{d}z \equiv \int_{z}^{1} \mathrm{d}z, \quad \int^{-} \mathrm{d}z \equiv \int_{0}^{z} \mathrm{d}z;$$

the $C_1^{\pm}(\tau)$ are the arbitrary 'constants'. It has been assumed in (13) that the surface wave, and also η_n^-, decay as $\xi \to +\infty$, i.e. there is no disturbance ahead of the wave. Of course, it is clear that we can expect particularly simple relations between $\eta_n = \eta_n^+$ and η_n^-, even when a critical layer is present.

The importance of quoting the solutions (11) and (13) is not so much in the form they take – although that may be of some interest – but rather in the expansions they generate as $z \to z_c$. If we set $z = z_c + \epsilon^{\frac{1}{2}} Z$, and then expand for $\epsilon \to 0^+$, we shall obtain the ϵ-dependence of the terms that must be present in the expansion of the inner (critical layer) region, as well as the required matching conditions. However, before we turn to this detailed aspect of the limiting behaviour we can note that $\psi_0^{\pm} \to \eta_0^{\pm}/F_c'$ as $z \to z_c$, which confirms the nature of the breakdown of the outer expansion. Furthermore, $\psi_1^{\pm}$ is unbounded as $z \to z_c$, emphasizing still more strongly the need for the critical layer.

The expansion of $p_0^{\pm} + \epsilon p_1^{\pm}$ and $\psi_0^{\pm} + \epsilon \psi_1^{\pm}$, as $z \to z_c$ is quite involved, particularly for the stream function. The complete expansion for $\psi_0^{\pm} + \epsilon \psi_1^{\pm}$, up to and including $O(\epsilon^{\frac{3}{2}})$, is given in the Appendix. We shall present here a somewhat reduced and simplified version which retains the salient features. The corresponding expansion for the pressure is quite straightforward, yielding

$$p_0^{\pm} + \epsilon p_1^{\pm} = \eta_0^{\pm} + \epsilon(\eta_1^{\pm} \pm K^{\pm} \eta_{0\xi\xi}^{\pm}) + o(\epsilon^{\frac{3}{2}}), \tag{14}$$

for $z = z_c + \epsilon^{\frac{1}{2}} Z$, as $\epsilon \to 0^+$ (and the $K^{\pm}$ are given below): the pressure perturbation, to this order, does not vary with Z. For the stream function we obtain

$$\begin{aligned} \psi_0^{\pm} + \epsilon \psi_1^{\pm} = {} & \frac{\eta_0^{\pm}}{F_c'} + \gamma_1 \epsilon^{\frac{1}{2}} \ln \epsilon + \epsilon^{\frac{1}{2}}(\gamma_2 - Z F_c' \not{I}^{\pm} \eta_0^{\pm}) + \gamma_3 \epsilon (\ln \epsilon)^2 + \gamma_4 \epsilon \ln \epsilon \\ & + \epsilon \left(\gamma_5 + \frac{\eta_1^{\pm}}{F_c'} \pm \frac{K^{\pm}}{F_c'} \eta_{0\xi\xi}^{\pm} \right) + \gamma_6 \epsilon^{\frac{3}{2}} (\ln \epsilon)^2 + \gamma_7 \epsilon^{\frac{3}{2}} \ln \epsilon \\ & + \epsilon^{\frac{3}{2}} \left[\gamma_8 + Z F_c' \left\{ -\not{I}^{\pm} \eta_1^{\pm} \pm 2 I_3^{\pm} \int_{\xi}^{\infty} \eta_{0\tau}^{\pm} \, \mathrm{d}\xi \right. \right. \\ & \left. \left. + (K^{\pm} I_2^{\pm} - L^{\pm}) \eta_{0\xi\xi}^{\pm} \pm \tfrac{3}{2} I_4^{\pm} (\eta_0^{\pm})^2 \right\} \right] + O(\epsilon^{\frac{3}{2}}), \end{aligned} \tag{15}$$

where the $\gamma_i = \gamma_i(Z, \xi, \tau)$ (for $i = 1, \ldots, 8$) can be deduced by comparing (15) with the full details in the Appendix. Only terms that differ above and below are recorded in (15), and then only the first time they occur. (For example, terms in both $\eta_0^{\pm}$ and $\not{I}^{\pm} \eta_0^{\pm}$ appear through the γ_i.) The notation adopted in (14) and (15) involves defining various integrals:

$$K^+ = \int_{z_c}^{1} F^2 I^+ \,\mathrm{d}z, \quad K^- = \int_{0}^{z_c} F^2 I^- \,\mathrm{d}z, \quad L^+ = \int_{z_c}^{1} F^2 I^+ (1 - I^+) \,\mathrm{d}z, \quad L^- = \int_{0}^{z_c} F^2 (I^-)^2 \,\mathrm{d}z, \tag{16}$$

and $\not{I}^{\pm}$, I_n^+, I_n^- are the finite parts (as $z \to z_c$) of $I^{\pm}$, $\int_z^1 F^{-n} \,\mathrm{d}z$, $\int_0^z F^{-n} \,\mathrm{d}z$, respectively. Thus we can write, for example,

$$\not{I}^- = I_2^- \quad \text{and} \quad \not{I}^+ = 1 - I_2^+,$$

and we note that all the finite parts are defined in the conventional way, e.g.

$$\int_z^1 F^{-2}\,dz = \frac{(z-z_c)^{-1}}{(F'_c)^2} + \frac{F''_c}{(F'_c)^3}\ln(z-z_c) + I_2^+ + O(1),$$

as $z \to z_c^+$. The integrals $K^{\pm}$, $L^{\pm}$ are clearly finite even though they involve $I^{\pm}$.

The expansion of the stream function given in (15) is a generalization of the corresponding results obtained by other authors, in particular Benney & Bergeron (1969) and Davis (1969). The form of (15) derived here is valid for an unsteady, and as yet unknown, nonlinear surface profile. This expansion, however, agrees with the general pattern described by Benney & Maslowe (1975). We can now use the limited information given in (15), together with an observation developed in the next paragraph, to determine the equation which describes the surface profile function, $\eta(\xi,\tau)$, to leading order. If we assume that the critical-layer solutions exist and are symmetric about $Z = 0$ (see §4), then the expansions of ψ^+ and ψ^-, as $z \to z_c$, must be identical since these expansions constitute matching conditions for the critical-layer solution. This constraint applied to the expansion (15), and more generally to the expansion given in the Appendix, yields the following identities:

$$\left.\begin{aligned} &\eta_0 = \eta_0^+ = \eta_0^-; \quad I^+ = I^- \quad \text{or} \quad I_2^+ + I_2^- = 1; \\ &\eta_1^+ + K^+\eta_{0\xi\xi}^+ = \eta_1^- - K^-\eta_{0\xi\xi}^-; \\ &2I_3^+\int_\xi^\infty \eta_{0\tau}^+\,d\xi + (K^+I_2^+ - L^+)\,\eta_{0\xi\xi}^+ + \tfrac{3}{2}I_4^+(\eta_0^+)^2 - I^+\eta_1^+ \\ &\quad = -2I_3^-\int_\xi^\infty \eta_{0\tau}^-\,d\xi + (K^-I_2^- - L^-)\,\eta_{0\xi\xi}^- - \tfrac{3}{2}I_4^-(\eta_0^-)^2 - I^-\eta_1^-. \end{aligned}\right\} \tag{17}$$

That there can be no other contributions to the terms given in (15), and used in (17), from the expansion of higher-order terms ($\epsilon^n\psi_n^{\pm}$, $n \geqslant 2$) is easily seen. The first and third equations that appear in (17) arise from the terms $O(1)$, $O(\epsilon)$, respectively; the second and fourth equations are generated by terms $O(\epsilon^{\frac{1}{2}}Z)$, $O(\epsilon^{\frac{3}{2}}Z)$ respectively. However, later terms in the expansion will produce, for example, terms of the form $O(\epsilon^n)$ and $O(\epsilon^{n+\frac{1}{2}}Z)$, $n \geqslant 2$: the identities presented in (17) are therefore complete and require no further amendment from terms unknown.

From (17) it is clear that the Burns integral condition for the linearized propagation speed c is replaced by

$$I_2^+ + I_2^- = 1, \tag{18}$$

which is just the finite part of the original condition (see Velthuizen & van Wijngaarden 1969). Also we see that the relation between η_1^+ and η_1^- is precisely that required to make the pressure continuous across $z = z_c$, on the basis of the outer solution alone, to this order: see (14). The equation defining $\eta_0(\xi,\tau)$ ($= \eta_0^+$) is now obtained from the fourth equation in (17), after incorporating the previous three identities, as

$$-2(I_3^+ + I_3^-)\,\eta_{0\tau} + 3(I_4^+ + I_4^-)\,\eta_0\,\eta_{0\xi} + (K^+ + L^- - L^+)\,\eta_{0\xi\xi\xi} = 0, \tag{19}$$

where a differentiation with respect to ξ has also been performed. The coefficient $K^+ + L^- - L^+$ turns out to be just the finite part of the integral J (see (3)). In other words, (19) is the KdV equation for waves over an arbitrary shear, but evaluated in terms of finite parts (cf. Redekopp 1977; Maslowe & Redekopp 1980).

Although the KdV equation, (19), admits an exact nonlinear periodic wave solution (the cnoidal wave), strictly speaking such a choice of solution cannot be adopted here since we have already incorporated the assumption that $\eta_0 \to 0$ as $\xi \to \infty$. Our concern is with unsteady solutions which may evolve into a solution with a number of maxima at one instant in time. For such solutions we require that $\eta_0 \to 0$ as $\xi \to +\infty$ (or $\xi \to -\infty$); in fact, the n-soliton solution satisfies $\eta_0 \to 0$ as $|\xi| \to \infty$, (for all $\tau < \infty$), as do all solutions with initial data on compact support. We are now in a position to investigate the consequences for the critical-layer region, as the surface wave evolves according to the KdV equation (19).

4. The critical layer region

The equations to be used in the description of the critical layer are given in (6), and the expansion (15) (see also the Appendix) provides the appropriate matching conditions as $Z \to \pm\infty$. If we introduce the stream function $\Psi(\xi, \tau, Z; \epsilon)$ then the pressure, P, and Ψ, are to be expanded as

$$\left.\begin{aligned} P &= P_0 + \epsilon P_1 + o(\epsilon); \\ \Psi &= \Psi_0 + \epsilon^{\frac{1}{2}}(\ln \epsilon \Psi_1 + \Psi_2) + \epsilon((\ln \epsilon)^2 \Psi_3 + \ln \epsilon \Psi_4 + \Psi_5) + o(\epsilon), \end{aligned}\right\} \tag{20}$$

both valid as $\epsilon \to 0^+$. Although more information is available concerning the form of these inner expansions (see (14) and (15)) we shall examine in detail only terms as far as $O(\epsilon)$. The asymptotic solution prescribed in (20) is to be determined by imposing the continuity of (a) pressure, (b) stream function and (c) particle velocity, at every point within the critical layer. Specifically, these conditions will apply on the separating streamline, although presumably we must allow a discontinuity in vorticity across this line. The separating streamline will, at the initial time, isolate streamlines which are defined for $-\infty < \xi < \infty$ with $\mathrm{d}Z/\mathrm{d}\xi$ finite everywhere from those which turn around and return whence they have come (and therefore these have $\mathrm{d}Z/\mathrm{d}\xi$ infinite at one point). At later times the separating streamlines may confine regions with closed streamlines.

The leading-order problem obtained from (6) and (20) produces a slight generalization of the classical stream function usually associated with the Kelvin cats'-eyes pattern. We have that

$$\left.\begin{aligned} P_{0Z} &= 0; \\ ZF_c' \Psi_{0\xi Z} - F_c' \Psi_{0\xi} + \Psi_{0Z} \Psi_{0\xi Z} - \Psi_{0\xi} \Psi_{0ZZ} + P_{0\xi} &= 0, \end{aligned}\right\} \tag{21}$$

and so $P_0 = \eta_0 (= \eta_0^+) = \eta_0^-$ if the pressure is to be continuous and to match to (14); also see (17). The equation for Ψ_0 can be expressed as

$$\hat{\Psi}_{0ZZ} = f(\hat{\Psi}_0) \quad \text{where } \hat{\Psi}_0 = \tfrac{1}{2} Z^2 F_c' + \Psi_0,$$

and $f(.)$ is an arbitrary function. Now at $\tau = 0$, and in fact until a second peak is formed, all the streamlines extend to infinity: no closed streamlines exist. At infinity we prescribe the undisturbed shear profile, i.e. $\hat{\Psi}_{0ZZ} \to F_c'$ as $|\xi| \to \infty$ (since $\eta_0 \to 0$ as $|\xi| \to \infty$), and the matching condition requires that $\Psi_0 \to \eta_0 / F_c'$ as $|Z| \to \infty$, whence

$$\Psi_0 = \frac{\eta_0}{F_c'} \quad \text{or} \quad \hat{\Psi}_0 = \tfrac{1}{2} Z^2 F_c' + \frac{\eta_0}{F_c'}, \tag{22}$$

setting $\Psi_0 = 0$ for $\eta_0 = 0$. The dominant representation for the streamlines is therefore

$$\tfrac{1}{2} Z^2 + \zeta = \zeta_0, \quad \zeta = \frac{\eta_0}{F_c'^2} \geqslant 0, \tag{23}$$

where ζ_0 is a constant. If ζ is determined from the initial wave profile which has a single maximum, $\zeta = \zeta_m$, then $\zeta_0 = \zeta_m$ will describe the separating streamlines. The streamlines interior are given by $\zeta_m > \zeta_0 > 0$; those exterior are given by $\zeta_0 > \zeta_m$. Of course, the perturbation stream function, and the total stream function, are defined for all Z, as are all the derivatives of Ψ_0 (and $\hat{\Psi}_0$): see (22). In consequence the vorticity is continuous throughout the critical-layer region, to leading order. Any particular streamline, for which ζ_0 is fixed by choosing an appropriate Z $(= \pm(2\zeta_0)^{\frac{1}{2}})$ as $\xi \to \pm\infty$, will now evolve (as τ varies) according to (23) (with $\zeta = \eta_0(\xi, \eta)/F_c'^2$, where η_0 satisfies the KdV equation, (19)). On the other hand, the separating streamline itself will also evolve as τ varies, since the maximum value of ζ (from the maximum of η_0) is a function of τ,

$$\tfrac{1}{2}Z^2 + \zeta = \zeta_m(\tau) \quad (\geqslant 0). \tag{24}$$

However, it must be remembered that (24) describes different streamlines at different times (cf. (23)): this streamline at any instant passes through the points $Z = \pm(2\zeta_m)^{\frac{1}{2}}$, $\xi \to \pm\infty$. Nevertheless it is the separating streamlines given by (24) that we shall require when the continuity conditions at higher order, and for all τ, are imposed.

The equation for $\Psi_1(\xi, \tau, Z)$ is

$$Z\Psi_{1\xi Z} - \Psi_{1\xi} - \zeta_\xi \Psi_{1ZZ} = 0, \tag{25}$$

which has the general solution

$$\Psi_1 = \int^Z \int^Z \omega_1(\tfrac{1}{2}Z^2 + \zeta, \tau)\, dZ\, dZ + \alpha_1 Z + \beta_1,$$

where ω_1 is the (arbitrary) vorticity contribution, and $\alpha_1(\xi, \tau)$, $\beta_1(\xi, \tau)$ are arbitrary functions (associated with the limits on the double integral). These three unknown functions will (in the most general problem) differ above, below and inside the separating streamlines. Now the total vorticity outside the separating streamlines is $-F_c' + O(\epsilon^{\frac{1}{2}})$ as $\xi \to +\infty$ and so $\omega_1 \to 0$ as $\xi \to +\infty$, for all Z. Hence $\omega_1 \equiv 0$, and then to match as $|Z| \to \infty$ we must set $\alpha_1 = \frac{1}{2}F_c''\zeta$; we choose $\beta_1 = 0$. The continuity of Ψ_1 and Ψ_{1Z} across $Z = \pm(2(\zeta_m - \zeta))^{\frac{1}{2}}$ then requires that

$$\Psi_1 = \tfrac{1}{2}F_c''\zeta Z, \tag{26}$$

everywhere: the vorticity inside is therefore uniquely determined. (The choice of zero constants in both (22) and (26) means that the matching back to the expansion for $\psi_0^\pm + \epsilon\psi_1^\pm$ yields $C_0^\pm = 0$.) The total stream function now takes the form

$$\tfrac{1}{2}Z^2 + \zeta + \epsilon^{\frac{1}{2}} \ln \epsilon \left(\frac{1}{2}\frac{F_c''}{F_c'}\right)\zeta Z, \tag{27}$$

to this order; the streamlines are given by setting expression (27) equal to ζ_0, and the separating streamlines are obtained (to this same order) by using ζ_m (the maximum value as defined earlier).

The next three terms in the critical-layer expansion are obtained by using similar arguments, although the matching requirements together with the continuity conditions now imply that Ψ_2 and Ψ_4 contribute a discontinuity in vorticity across the separating streamlines. If $Z = Z_s(\xi, \tau; \epsilon)$ is the separating streamline, then Ψ and Ψ_Z are to be continuous on $Z = Z_s$ if the stream function and particle velocity are to be continuous on this line. Noting that $Z_s \sim \pm Z_0$ as $\epsilon \to 0^+$, where $Z_0 = (2(\zeta_m - \zeta))^{\frac{1}{2}}$, the continuity conditions are expanded about $Z = \pm Z_0$ to yield Ψ_n, Ψ_{nZ} $(n = 0, \ldots, 3)$, Ψ_4 and $(\Psi_{4Z} - \frac{1}{2}(F_c''/F_c')\zeta\Psi_{2ZZ})$ continuous across $Z = \pm Z_0$. The details in the derivation of the solutions for Ψ_2, Ψ_3 and Ψ_4 will be omitted; the methods will, however, be explained with reference to the solution for Ψ_5. We obtain, after a little analysis,

$$\Psi_2 = -\tfrac{1}{6}Z^3F_c''+\alpha_2 Z+\begin{cases} \pm F_c''\int_{\pm Z_0}^{Z}\left(\int_{\pm Z_0}^{Z}(Z^2+2\zeta)^{\frac{1}{2}}\,\mathrm{d}Z\right)\mathrm{d}Z, & |Z|\geqslant Z_0 \\ 0, & |Z|\leqslant Z_0; \end{cases} \tag{28}$$

$$\Psi_3 = \frac{1}{8}\frac{F_c''^2}{F_c'}\zeta^2 \quad (\forall Z); \tag{29}$$

$$\Psi_4 = \frac{1}{2}\frac{F_c''}{F_c'}\left(\alpha_2\zeta-\int_{\xi}^{\infty}\zeta_\tau\,\mathrm{d}\xi\right)+\begin{cases} \pm\frac{1}{2}\frac{F_c''^2}{F_c'}\zeta\int_{\pm Z_0}^{Z}(Z^2+2\zeta)^{\frac{1}{2}}\,\mathrm{d}Z, & |Z|\geqslant Z_0 \\ 0, & |Z|\leqslant Z_0, \end{cases} \tag{30}$$

where $$\alpha_2 = F_c''[\ln((2\zeta_m)^{\frac{1}{2}}+Z_0)+1-\ln 2]\,\zeta+\tfrac{1}{2}F_c''Z_0(2\zeta_m)^{\frac{1}{2}}-F_c'I_2^{\pm}\,\eta_0. \tag{31}$$

(The signs are ordered: '+' in $Z > Z_0$ and '−' in $Z < -Z_0$, and $I_2^+ = I_2^-$ since α_2 must be continuous in $-Z_0 < Z < Z_0$; see (17).) The integrals appearing in (28) and (30) can be evaluated in terms of elementary functions (which greatly aids the matching), but it is much neater to leave them in this form here. The solutions so far obtained show that Ψ_0 is even (in Z), Ψ_1 odd, Ψ_2 odd, Ψ_3 even, Ψ_4 even, and similar conclusions apply to all the Ψ_n; Ψ_5 turns out to be even. This confirms the symmetry property used in the previous paragraph.

If we use the details given above, we find that the separating streamline becomes

$$Z_s \sim \pm Z_0-\epsilon^{\frac{1}{2}}\ln\epsilon\left(\frac{1}{2}\frac{F_c''}{F_c'}\right)\zeta+\frac{\epsilon^{\frac{1}{2}}}{F_c'}\left(-\alpha_2+\frac{1}{Z_0}\int_{\xi}^{\xi_m}Z_{0\tau}\,\mathrm{d}\xi\right), \tag{32}$$

as $\epsilon\to 0^+$, where $\xi = \xi_m(\tau)$ is the path of the point of largest amplitude (i.e. $\zeta_m(\tau) = \zeta(\xi_m,\tau)$).† It is easily confirmed that there is no non-uniformity in this expansion as $\xi\to\xi_m$, even though $Z_0(\xi_m,\tau) = 0$. An interesting observation is that the separating streamline is seen to be asymmetric now that the higher-order terms have been included (a point also made by Moore & Saffman 1982). It turns out that the separating streamline, given here correct at $O(\epsilon^{\frac{1}{2}})$, is of sufficient accuracy to allow the complete determination of Ψ_5 (which we shall consider in more detail later). The solutions (29) and (30), for Ψ_3 and Ψ_4 respectively, match with the outer solutions to the extent that there are no contributions of $O(\epsilon(\ln\epsilon)^2)$ or $O(\epsilon\ln\epsilon)$ to the arbitrary constant. However the solution for Ψ_2, apart from the terms ensuring matching to the $O(\epsilon^{\frac{1}{2}})$ term (see Appendix), incorporates the additional term $\pm\frac{1}{3}(2\xi_m)^{\frac{3}{2}}$ as $Z\to\pm\infty$. In consequence, to match, the outer expansion must now read

$$\psi \sim \psi_0^{\pm}\pm\epsilon^{\frac{1}{2}}\tfrac{1}{3}(2\zeta_m)^{\frac{3}{2}}+\epsilon\psi_1^{\pm} \quad \text{as } \epsilon\to 0^+; \tag{33}$$

the additional term, being a function only of τ, automatically satisfies the governing equations at $O(\epsilon^{\frac{1}{2}})$. There is no $O(\epsilon^{\frac{1}{2}})$ contribution to the pressure. This extra term appearing in (33) is the correction necessary in the outer region in order to accommodate the distortion of the streamlines within the critical layer. An analogous correction term was also found by Moore & Saffman (1982); a more general $O(\epsilon^{\frac{1}{2}})$ addition to the outer stream function (following Haberman 1972) will be mentioned in §5.

† Note that ξ_m is well defined for a single-peaked initial profile which evolves into n solitons (plus oscillatory tail). The occurrence of twin (equal) peaks, however, requires minor modifications to permit an interpretation of (32).

Although we have chosen not to dwell upon the details of the derivation of (28)–(30) (but see later), we shall indicate the contribution these terms make to the total vorticity. Since the expansion (20) has been determined to $O(\epsilon \ln \epsilon)$, the contribution from Ψ_2, Ψ_3 and Ψ_4 to the vorticity will be

$$-(\epsilon^{\frac{1}{2}}\Psi_2+\epsilon(\ln\epsilon)^2\,\Psi_3+\epsilon\,\ln\epsilon\Psi_4)_{ZZ}$$

$$=-\epsilon^{\frac{1}{2}}\left[-F_c'' Z\pm\left(F_c''(Z^2+2\zeta)^{\frac{1}{2}}+\epsilon^{\frac{1}{2}}\ln\epsilon\,\frac{F_c''^2}{F_c'}\frac{\zeta Z}{(Z^2+2\zeta)^{\frac{3}{2}}}\right)\right],\quad |Z|>Z_0, \tag{34}$$

and just $\epsilon^{\frac{1}{2}}F_c'' Z$ in $|Z|<Z_0$. In other words, the total vorticity is

$$-F_c'-\epsilon^{\frac{1}{2}}F_c'' Z+o(\epsilon\,\ln\epsilon),$$

as $\zeta\to 0$ (i.e. $|\xi|\to\infty$), in the exterior regions (above and below the separating streamlines). The region interior then has a total vorticity (uniquely determined by using the continuity conditions on $Z=\pm Z_0$) of $-F_c'$, to this order, no matter what the form of the surface wave. Thus the region which can support closed streamlines is one of constant vorticity (at least to $o(\epsilon\ln\epsilon)$): this is consistent with the Prandtl–Batchelor theorem. The variation in vorticity required in the exterior region to maintain the passage of the wave is described explicitly in (34), and for consistency in our problem this must pertain at $\tau=0$.

We shall now examine more closely the derivation of the solution for Ψ_5; the same methods have already been employed to find Ψ_2, Ψ_3 and Ψ_4. The following discussion should therefore clarify the points omitted in our presentation of the solutions (28)–(30).

The problem for Ψ_5 and P_1 can be written as

$$\left.\begin{aligned}ZF_c'\,\Psi_{5\xi Z}-F_c'\,\Psi_{5\xi}-\frac{\eta_{0\xi}}{F_c'}\,\Psi_{5ZZ}+\tfrac{1}{2}Z^2F_c''\,\Psi_{2\xi Z}-\tfrac{1}{2}Z^2\,\frac{F_c'''}{F_c'}\,\eta_{0\xi}\qquad&\\ -ZF_c''\,\Psi_{2\xi}+\Psi_{2Z}\,\Psi_{2\xi Z}-\Psi_{2\xi}\,\Psi_{2ZZ}+\Psi_{2Z\tau}+P_{1\xi}=0,&\\ \text{with}\qquad P_{1Z}=0,&\end{aligned}\right\} \tag{35}$$

where we have already made use of $\Psi_0=\eta_0/F_c'$; $\Psi_2(\xi,\tau,Z)$ is given in (28). The pressure P_1 is then continuous and matches with (14) if

$$P_1=\eta_1^+ + K^+\eta_{0\xi\xi}=\eta_1^- - K^-\eta_{0\xi\xi}, \tag{36}$$

and again see (17). The solution for Ψ_5 valid in the exterior region of the critical layer can now be obtained from (35) by first substituting for P_1 and $\Psi_2^\pm$: after some analysis we find that

$$\begin{aligned}\Psi_5^\pm=&-\frac{F_c'''}{24}Z^4+\frac{1}{F_c'}\left(\eta_1^\pm\pm K^\pm\eta_{0\xi\xi}+\tfrac{1}{2}\alpha_2^2-\int_\xi^\infty\alpha_{2\tau}\,d\xi\right)\pm\alpha_2\,\frac{F_c''}{F_c'}\int_{\pm Z_0}^{Z}(Z^2+2\zeta)^{\frac{1}{2}}\,dZ\\ &+\frac{F_c''}{F_c'}\int_{\pm Z_0}^{Z}\int_{\pm Z_0}^{Z}\frac{1}{(Z^2+2\zeta)^{\frac{1}{2}}}\int_\xi^\infty\frac{\hat{\zeta}_\tau}{(Z^2+2\zeta-2\hat{\zeta})^{\frac{1}{2}}}\,d\hat{\xi}\,dZ\,dZ+\frac{F_c''}{F_c'}(2\zeta_m)^{\frac{1}{2}}\int_\xi^\infty Z_{0\tau}\,d\xi\\ &+\frac{F_c''^2}{F_c'}\int_{\pm Z_0}^{Z}\int_{\pm Z_0}^{Z}\frac{1}{(Z^2+2\zeta)^{\frac{3}{2}}}\left(\int_{\pm Z_0}^{Z}\int_{\pm Z_0}^{Z}(Z^2+2\zeta)^{\frac{1}{2}}\,dZ\,dZ\right)dZ\,dZ\\ &+\int_{\pm Z_0}^{Z}\int_{\pm Z_0}^{Z}\omega_5^\pm(Z^2+2\zeta,\tau)\,dZ\,dZ+\alpha_5^\pm(\xi,\tau)\,Z+\beta_5^\pm(\tau),\end{aligned} \tag{37}$$

where $\omega_5^{\pm}$, $\alpha_5^{\pm}$ and $\beta_5^{\pm}$ are arbitrary functions, and $\zeta = \zeta(\xi, \tau)$. To determine $\omega_5^{\pm}$ it is necessary to consider the $O(\epsilon)$ contribution to the vorticity; this takes the form

$$-(\tfrac{1}{2}Z^2F_c''' + \Psi_{5ZZ}) = -\left(\pm\alpha_2^2 \frac{F_c''}{F_c'}\frac{Z}{(Z^2+2\zeta)^{\frac{1}{2}}} + \frac{F_c''}{F_c'}\frac{1}{(Z^2+2\zeta)^{\frac{1}{2}}}\int_{\xi}^{\infty}\frac{\hat{\zeta}_\tau}{(Z^2+2\zeta-2\hat{\zeta})^{\frac{1}{2}}}\,\mathrm{d}\hat{\xi} + \frac{F_c''^2}{F_c'}\frac{1}{(Z^2+2\zeta)^{\frac{1}{2}}}\int_{\pm Z_0}^{Z}\int_{\pm Z_0}^{Z}(Z^2+2\zeta)^{\frac{1}{2}}\,\mathrm{d}Z\,\mathrm{d}Z + \omega_5^{\pm}\right). \quad (38)$$

Moreover, the terms in (38) must approach $-\frac{1}{2}Z^2F_c'''$ as $\xi \to +\infty$ (where $\eta_0, \zeta \to 0$), which requires that

$$\omega_5^{\pm}(X, \tau) = \frac{1}{2}\left(F_c''' - \frac{1}{3}\frac{F_c''^2}{F_c'}\right)X - \frac{1}{3}\frac{F_c''^2}{F_c'}\frac{(2\zeta_m)^{\frac{3}{2}}}{X^{\frac{1}{2}}}; \quad X = Z^2+2\zeta, \quad (39)$$

being the same both above and below the separating streamlines. The discussion of the exterior region is completed by matching (37), as $|Z| \to \infty$, to the $O(\epsilon)$ terms given in the Appendix. Although a lengthy business, this does confirm that (37) is, indeed, the correct form, with $\omega_5^{\pm}$ given by (39); furthermore, the absence of any term linear in Z, as $|Z| \to \infty$, is possible only if

$$\alpha_5^{\pm} = \pm Z_0(2\zeta+\zeta_m)\left(\tfrac{1}{3}F_c''' - \frac{F_c''^2}{F_c'}\right) \mp \frac{F_c''}{F_c'}\int_{\xi}^{\infty}\frac{\hat{\zeta}_\tau}{(2\zeta)^{\frac{1}{2}}}\mathscr{F}\left(\phi, \left(\frac{\hat{\zeta}}{\zeta}\right)^{\frac{1}{2}}\right)\mathrm{d}\hat{\xi}, \quad (40)$$

where $\mathscr{F}(\phi, k)$ is the incomplete elliptic integral of the first kind, with $\tan\phi = (2\zeta)^{\frac{1}{2}}/Z_0$. Finally, the term $\beta_5^{\pm}(\tau)$ is simply related to $C_1^{\pm}(\tau)$: to match we must have

$$C_1^{\pm} = \beta_5^{\pm} + \tfrac{1}{2}\zeta_m^2\left(F_c''' - 3\frac{F_c''^2}{F_c'}\right). \quad (41)$$

The interior region of the critical layer is described by equations (35), with P_1 already determined as (36), and Ψ_2 expressed in the form valid for $|Z| \leqslant Z_0$ (see (28)). The integration of (35) yields the general solution for Ψ_5^{i} (defined in $|Z| \leqslant Z_0$) as

$$\Psi_5^{i} = -\frac{F_c'''}{24}Z^4 + \frac{1}{F_c'}\left(\eta_1^{\pm} \pm K^{\pm}\eta_{0\xi\xi} + \tfrac{1}{2}\alpha_2^2 - \int_{\xi}^{\infty}\alpha_{2\tau}\,\mathrm{d}\xi\right) + \int_0^Z\int_0^Z \omega_5^{i}(\tfrac{1}{2}Z^2+\zeta, \tau)\,\mathrm{d}Z\,\mathrm{d}Z + \alpha_5^{i}Z - \int_0^{\zeta}\omega_5^{i}(X, \tau)\,\mathrm{d}X + \beta_5^{i}, \quad (42)$$

where ω_5^{i}, $\alpha_5^{i}(\xi, \tau)$ and $\beta_5^{i}(\tau)$ are arbitrary functions. (Solution (42) is no more than solution (37) with certain terms in the particular integral omitted and the limits on the double integral specified.) The arbitrary functions are now chosen to ensure that Ψ (to this order) satisfies the continuity conditions on $Z = Z_s(\xi, \tau; \epsilon)$. The continuity of Ψ, at this order, requires that Ψ_5 be continuous on $Z = \pm Z_0$; the continuity of Ψ_Z requires the continuity of $(\Psi_{5Z} + Z_2\,\Psi_{2ZZ})$, where we have written

$$Z_s \sim \pm Z_0 + \epsilon^{\frac{1}{2}}\ln\epsilon Z_1 + \epsilon^{\frac{1}{2}}Z_2, \quad \epsilon \to 0^+,$$

for simplicity: see (32). These two conditions yield

$$\frac{F_c'''}{F_c'}(2\zeta_m)^{\frac{1}{2}}\int_{\xi}^{\infty}Z_{0\tau}\,\mathrm{d}\xi \pm \alpha_5^{\pm}Z_0 + \beta_5^{\pm} = \int_0^{Z_0}\int_0^{Z}\omega_5^{i}\,\mathrm{d}Z\,\mathrm{d}Z - \int_0^{\zeta}\omega_5^{i}(X, \tau)\,\mathrm{d}X + \beta_5^{i}, \quad (43)$$

and

$$\frac{F_c''}{F_c'}\frac{(2\zeta_m)^{\frac{1}{2}}}{Z_0}\int_{\xi}^{\xi_m}Z_{0\tau}\,\mathrm{d}\xi \pm \alpha_5^{\pm} = \int_0^{Z_0}\omega_5^{i}\,\mathrm{d}Z \quad (44)$$

respectively, with $\alpha_5^{\mathrm{i}} = 0$ and $\beta_5^+ = \beta_5^-$ (and we note that $\alpha_5^+ = -\alpha_5^-$: see (40)). It is convenient at this stage to write the vorticity contribution, ω_5^{i}, as the sum of two parts,

$$\omega_5^{\mathrm{i}} = \left(F_{\mathrm{c}}''' - 3\frac{F_{\mathrm{c}}''^2}{F_{\mathrm{c}}'}\right)(\tfrac{1}{2}Z^2 + \zeta) + \Omega_5(\tfrac{1}{2}Z^2 + \zeta, \tau), \tag{45}$$

whence (43) and (44) yield

$$\beta_5^{\pm} - \beta_5^{\mathrm{i}} = \frac{1}{2}\left(3\frac{F_{\mathrm{c}}''^2}{F_{\mathrm{c}}'} - F_{\mathrm{c}}'''\right)\zeta_{\mathrm{m}}^2 - \frac{F_{\mathrm{c}}''}{F_{\mathrm{c}}'}(2\zeta_{\mathrm{m}})^{\frac{1}{2}}\int_{\xi_{\mathrm{m}}}^{\infty} Z_{0\tau}\,\mathrm{d}\xi - \int_0^{\zeta_{\mathrm{m}}} \Omega_5(X,\tau)\,\mathrm{d}X, \tag{46}$$

and
$$\int_0^{Z_0} \Omega_5\,\mathrm{d}Z = \frac{F_{\mathrm{c}}''}{F_{\mathrm{c}}'}\frac{(2\zeta_{\mathrm{m}})^{\frac{1}{2}}}{Z_0}\int_{\xi}^{\xi_{\mathrm{m}}} Z_{0\tau}\,\mathrm{d}\xi - \frac{F_{\mathrm{c}}''}{F_{\mathrm{c}}'}\frac{1}{(2\zeta)^{\frac{1}{2}}}\int_{\xi}^{\infty} \hat{\zeta}_\tau\,\mathscr{F}\left(\phi, \left(\frac{\hat{\zeta}}{\zeta}\right)^{\frac{1}{2}}\right)\mathrm{d}\hat{\xi}. \tag{47}$$

The presentation of the complete solution of the critical layer, correct at $O(\epsilon)$, lacks only the solution of (47) for Ω_5. However, this final calculation is at least straightforward since (47) can be re-cast as Abel's integral equation (with degree $\frac{1}{2}$), although the actual form of the resulting solution is somewhat involved. It is worth observing that an Abel equation for the vorticity was also encountered by Varley & Blythe (1983) in their discussion of the critical layer; we shall return to this point later. Now, before we describe this solution in detail, it is instructive to first determine Ω_5 in the case of steady flow. In the context of our problem this will correspond only to the solitary-wave solution for the surface profile. If $\zeta = \zeta(\xi - C\tau)$, so that the solitary wave has an amplitude proportional to C, then we obtain

$$\int_0^{Z_0} \Omega_5\,\mathrm{d}Z = -C\frac{F_{\mathrm{c}}''}{F_{\mathrm{c}}'}\left\{(2\zeta_{\mathrm{m}})^{\frac{1}{2}}\left[\frac{Z_0(\hat{\zeta})}{Z_0(\zeta)}\right]_{\hat{\zeta}=\zeta}^{\hat{\zeta}=\zeta_{\mathrm{m}}} - \frac{1}{(2\zeta)^{\frac{1}{2}}}\left[E(\phi,k) - \left(1 - \frac{\hat{\zeta}}{\zeta}\right)\mathscr{F}(\phi,k) + \frac{Z_0(\zeta)}{(2\zeta)^{\frac{1}{2}}}\left(\left(1 - \frac{\hat{\zeta}}{\zeta_{\mathrm{m}}}\right)^{\frac{1}{2}} - 1\right)\right]_{\hat{\zeta}=\zeta}^{\hat{\zeta}=0}\right\},$$

where $E(\phi, k)$ is the incomplete elliptic integral of the second kind, and $k = (\hat{\zeta}/\zeta)^{\frac{1}{2}}$. This yields

$$\int_0^{Z_0} \Omega_5\,\mathrm{d}Z = C\frac{F_{\mathrm{c}}''}{F_{\mathrm{c}}'}Z_0 \tag{48}$$

and so $\Omega_5 = CF_{\mathrm{c}}''/F_{\mathrm{c}}'$, a constant. Such a simple solution for steady flow might encourage us to expect, after all, a fairly elementary solution for Ω_5 in general. However, this proves to be misleading: if we write (47) as

$$\frac{F_{\mathrm{c}}'}{F_{\mathrm{c}}''}\int_{\zeta}^{\zeta_{\mathrm{m}}} \frac{\Omega_5(X,\tau)}{(2(X-\zeta))^{\frac{1}{2}}}\,\mathrm{d}X = \frac{(2\zeta_{\mathrm{m}})^{\frac{1}{2}}}{Z_0}\int_{\xi}^{\xi_{\mathrm{m}}} Z_{0\tau}\,\mathrm{d}\xi - \frac{1}{(2\zeta)^{\frac{1}{2}}}\int_{\xi}^{\infty} \hat{\zeta}_\tau\,\mathscr{F}\left(\phi, \left(\frac{\hat{\zeta}}{\zeta}\right)^{\frac{1}{2}}\right)\mathrm{d}\hat{\xi}, \tag{49}$$

then the solution can be derived using standard techniques. After some manipulation this solution is expressed as

$$\begin{aligned}\frac{\pi}{\sqrt{2}}\frac{F_{\mathrm{c}}'}{F_{\mathrm{c}}''}\Omega_5(X,\tau) = {}& \frac{\Omega_{50}}{(\zeta_{\mathrm{m}} - X)^{\frac{1}{2}}} - \frac{(2\zeta_{\mathrm{m}})^{\frac{1}{2}}}{\zeta_{\mathrm{m}} - X}\int_X^{\zeta_{\mathrm{m}}} (\zeta - X)^{\frac{1}{2}}\left(\frac{\mathrm{d}\zeta_{\mathrm{m}}/\mathrm{d}\tau - \zeta_\tau}{\zeta_\xi}\right)_{\xi}\frac{\mathrm{d}\zeta}{\zeta_\xi} \\ &+ \int_X^{\zeta_{\mathrm{m}}}\left[\frac{(2\zeta_{\mathrm{m}})^{\frac{1}{2}}}{(\zeta_{\mathrm{m}} - \zeta)^{\frac{1}{2}}}\arctan\left(\frac{\zeta - X}{\zeta_{\mathrm{m}} - \zeta}\right)^{\frac{1}{2}} - \frac{\ln\left((\zeta^{\frac{1}{2}} + (\zeta_{\mathrm{m}})^{\frac{1}{2}})/(\zeta_{\mathrm{m}} - \zeta)^{\frac{1}{2}}\right)}{(2\zeta(\zeta - X))^{\frac{1}{2}}}\right]\frac{\zeta_\tau}{\zeta_\xi}\,\mathrm{d}\zeta \\ &- \tfrac{1}{2}\pi\frac{1}{(2\zeta_{\mathrm{m}})^{\frac{1}{2}}}\int_0^{\zeta_{\mathrm{m}}} \frac{1}{(\zeta_{\mathrm{m}} - \zeta)^{\frac{1}{2}}}\frac{\zeta_\tau}{\zeta_\xi}\,\mathrm{d}\zeta + \int_X^{\zeta_{\mathrm{m}}}\int_{\xi}^{\infty} \frac{\partial/\partial\hat{\tau}\,(E - \mathscr{F})}{(2\zeta(\zeta - X))^{\frac{1}{2}}}\,\mathrm{d}\hat{\xi}\,\mathrm{d}\zeta, \end{aligned} \tag{50}$$

where E and $\mathscr{F}$ are functions of ϕ and $k = (\hat{\zeta}/\zeta)^{\frac{1}{2}}$, and $\partial/\partial\hat{\tau}$ denotes the derivative with respect to the τ-dependence in $\hat{\zeta}$ only; $\Omega_{50}(\tau)$ is the limiting form of the right-hand side of (49) as $X \to \zeta_m$ or, equivalently, as $\xi \to \xi_m$,

$$\Omega_{50}(\tau) = (2\zeta_m)^{\frac{1}{2}} \frac{d\xi_m}{d\tau} - \frac{1}{(2\zeta_m)^{\frac{1}{2}}} \int_{\xi_m}^{\infty} \hat{\zeta}_\tau \, K\left(\left(\frac{\hat{\zeta}}{\zeta_m}\right)^{\frac{1}{2}}\right) d\hat{\xi}, \tag{51}$$

where K is the complete elliptic integral of the first kind. It is now an elementary exercise to check that (50) recovers the steady-state solution (for which $\Omega_{50} \equiv 0$), this calculation being aided by the fact that the integrals in (50) are over ζ, and $\zeta_\tau/\zeta_\xi = -C$ in the steady state.

This completes the presentation of the detailed results than can be obtained from our analysis. In particular we have demonstrated that the vorticity can be completely determined; this requires a specification of the initial state in order to fix Ψ_0 but for higher-order terms continuity conditions alone are involved. This slightly surprising result is essentially a consequence of restricting the analysis to long waves for which the leading order (Ψ_0) is sufficient to prescribe the dominant form of the separating streamline (across which continuity is applied). This important observation is made by Varley and Blythe (1983) in their study of the hydraulic equations (which are the ultimate long-wave equations). Since the form of the vorticity usually excites some interest for critical-layer flows, we shall conclude this section with a few observations. In fact, it turns out that the behaviour of the vorticity inside the separating streamlines is not as straightforward as one might have hoped.

The total vorticity† outside the separating streamlines can be written as

$$\begin{aligned} -\omega \sim F_c' &\pm \epsilon^{\frac{1}{2}} F_c''(X)^{\frac{1}{2}} \pm \epsilon \ln \epsilon \left(\frac{1}{2}\frac{F_c''^2}{F_c'}\right) \frac{\zeta Z}{(X)^{\frac{1}{2}}} \\ &+ \epsilon \left[\pm \alpha_2 \frac{F_c''}{F_c'} \frac{Z}{(X)^{\frac{1}{2}}} + \frac{1}{2}\left(F_c''' - \frac{1}{3}\frac{F_c''^2}{F_c'}\right) X - \frac{1}{3}\frac{F_c''^2}{F_c'} \frac{(2\zeta_m)^{\frac{3}{2}}}{(X)^{\frac{1}{2}}} \right. \\ &\left. + \frac{F_c''}{F_c'} \frac{1}{(X)^{\frac{1}{2}}} \int_{\xi}^{\infty} \frac{\hat{\zeta}_\tau}{(X - 2\hat{\zeta})^{\frac{1}{2}}} d\hat{\xi} + \frac{F_c''^2}{F_c'} \frac{1}{(X)^{\frac{1}{2}}} \int_{\pm Z_0}^{Z} \int_{\pm Z_0}^{Z} (X)^{\frac{1}{2}} dZ\, dZ \right], \end{aligned} \tag{52}$$

where $X = Z^2 + 2\zeta$, $\epsilon \to 0^+$ and the signs are ordered above/below. This is the required form for the maintenance of the wave motion, and must pertain at $\tau = 0$ to be consistent with our theory. It is clear that, for $\xi \to +\infty$, we recover the undisturbed vorticity distribution ahead of the wave

$$-\omega \sim F_c' + \epsilon^{\frac{1}{2}} Z F_c'' + \tfrac{1}{2}\epsilon Z^2 F_c''' \quad \text{as } \epsilon \to 0^+.$$

On the other hand, the total vorticity inside the separating streamline is

$$-\omega \sim F_c' + \epsilon \left[\frac{1}{2}\left(F_c''' - 3\frac{F_c''^2}{F_c'}\right) X + \Omega_5\right], \tag{53}$$

as $\epsilon \to 0^+$, where Ω_5 is given in (50). The vorticity is no longer constant, even though (53) is valid in the region which can support closed streamlines. However, this flow is not steady and so there is no conflict with the Prandtl–Batchelor theorem. If the flow were steady then

$$-\omega \sim F_c' + \epsilon \left[\left(F_c''' - 3\frac{F_c''^2}{F_c'}\right)(\tfrac{1}{2}Z^2 + \zeta) + C\frac{F_c''}{F_c'}\right], \tag{54}$$

† For convenience we shall give expressions for $-\omega$.

which is also not constant, but neither are there closed streamlines in this situation. Moreover we see that the vorticity is completely determined and not equal to $-F'_c$ as $\xi \to +\infty$: in particular, for example,

$$-\omega \sim F'_c + \epsilon\left[\frac{1}{2}\left(F'''_c - 3\frac{F''^2_c}{F'_c}\right)Z^2 - \frac{F''_c}{F'_c}\frac{1}{(2\zeta_m)^{\frac{1}{2}}}\int_{\xi_m}^{\infty} Z_{0_\tau}\,\mathrm{d}\xi\right], \tag{55}$$

for $\xi \to +\infty$ (since $\zeta \to 0$). It is easily confirmed that, in all regions, ω is constant on particles i.e. $\mathrm{D}\omega/\mathrm{D}t = 0$.

Finally, in order to determine the jump in vorticity across $Z = Z_s$, we require the behaviour of Ω_5 as $Z \to \pm Z_0$: but Ω_5 is singular in this limit, at least for unsteady flow for which $\Omega_{50} \neq 0$. In other words the asymptotic expansion for the vorticity is presumably not uniformly valid for $Z \in [-Z_0, Z_0]$, in the case of unsteady flows.

There are a number of avenues open to us which should lead to a more complete understanding of the nature of this singularity. The method we shall adopt is arguably the neatest and, in addition, it will furnish a direct comparison between our approach and that adopted by Varley & Blythe (1983). There should also be a fair measure of agreement between the results since our problem is essentially a perturbation of the hydraulic equations examined by Varley & Blythe. Furthermore, we can take the opportunity to check the form of the vorticity inside the separating streamlines. Consider the equation

$$\psi_Z\psi_{Z\xi} - \psi_\xi\psi_{ZZ} + q_\xi(\xi, \tau; \epsilon) = 0; \tag{56}$$

then $\psi_{ZZ} = -\hat{\omega}(\psi, \tau)$, where $\hat{\omega}$ is an arbitrary function, and if q is correct to $O(\epsilon)$ the total vorticity will be $\hat{\omega} + o(\epsilon)$. The choice

$$q = \eta_0 + \epsilon(\eta_1 + K^+\eta_{0\xi\xi}) - \tfrac{1}{2}\epsilon\ln\epsilon\, F''_c\int_\xi^\infty \zeta_\tau\,\mathrm{d}\xi - \epsilon\int_\xi^\infty \alpha_{2\tau}\,\mathrm{d}\xi,$$

ensures that we have the appropriate representation of

$$\int^\xi (p_\xi + \epsilon^{\frac{1}{2}}\psi_{Z\tau})\,\mathrm{d}\xi,$$

as $\epsilon \to 0^+$, valid inside the separating streamlines (see (6)), and the method of Varley & Blythe (1983) is applicable.

Now if $u = u_s^\pm$ on $Z = Z_s^\pm$ (the separating streamlines) and if $Z = Z_b$ on $Y = 0$ (where $u = 0$), then it can be shown that

$$\pm Z_s^\pm \mp Z_b = \frac{1}{2}\int_{2q}^{2q+(u_s^\pm)^2} \frac{(-\hat{\omega}^{-1})}{(s-2q)^{\frac{1}{2}}}\,\mathrm{d}s, \tag{57}$$

an integral equation for $\hat{\omega}$. Let us suppose that

$$\hat{\omega} \sim -F'_c + \epsilon\Omega; \tag{58}$$

then from (57), together with the continuity of particle velocities, we obtain

$$\int_\zeta^{\zeta_m} \frac{\Omega(z,\tau)}{(2(z-\zeta))^{\frac{1}{2}}}\,\mathrm{d}z = \mp\alpha_5^\pm - \frac{F''_c}{F'_c}\frac{(2\zeta_m)^{\frac{1}{2}}}{Z_0}\int_\xi^{\xi_m} Z_{0_\tau}\,\mathrm{d}\xi. \tag{59}$$

After a suitable choice of integration variable, (59) agrees in every respect with (44); we then have $\omega_5^{\mathrm{i}} = -\Omega$. This confirms the correctness of the vorticity distribution inside, but not close to, the separating streamlines.

In order to examine the behaviour of $\hat{\omega}$ near the boundaries $Z = Z_s^\pm$ it is necessary to allow the lower limit in (57) to approach the upper one i.e. $\xi \to \xi_m(\tau)$, at fixed ϵ. An initial study seems to indicate that

$$\hat{\omega} \sim -\epsilon^{\frac{1}{2}} \frac{1}{\sqrt{2}} \frac{F_c''}{F_c'} \frac{\Omega_{50}(\tau)}{(\zeta_m - X)^{\frac{1}{4}}} \quad \text{as } X \to \zeta_m^-, \quad (\epsilon \text{ fixed}). \tag{60}$$

Thus the existence of a singularity in the vorticity is reinforced: for unsteady flow ($\Omega_{50} \neq 0$) the separating streamlines are vortex sheets of strength $O(\epsilon^{\frac{1}{2}})$. For steady flow ($\Omega_{50} = 0$) there is only a discontinuity in vorticity across these streamlines. (The details underlying the above discussion can be found in Varley & Blythe (1983); it would appear that the integral equation (57) is worthy of further study.)

5. Discussion: the formation of a cat's-eye

This problem has demonstrated how the classical nonlinear critical layer can be extended to accommodate unsteady weakly nonlinear surface waves, corresponding to the analysis for Rossby waves as described by Redekopp (1977), and others. In particular we have seen that the vorticity is completely determined everywhere, without the need to invoke a viscous secularity condition, although the vorticity cannot be arbitrarily assigned initially. The restriction to long waves imposes a certain structure on the problem, as explained by Varley & Blythe (1983), which in turn fixes the vorticity inside the separating streamlines. Furthermore, we have shown that for unsteady motion the inclusion of a discontinuity in vorticity is not sufficient: it has been found that a weak vortex sheet must exist at the separating boundary. For steady flow, however, a jump in vorticity is the only ingredient. The jump is given by the difference between (52) and (54) evaluated on $Z = Z_s^{\pm}$, the dominant contribution being

$$[\omega] \sim \mp \epsilon^{\frac{1}{2}} F_c'' (2\zeta_m)^{\frac{1}{2}}, \text{ as } \epsilon \to 0^+,$$

where $[\omega]$ is the jump from inside to outside. Note that this jump is the same strength as the vortex sheet.

Let us now digress in order to see how the extension (for inviscid flows) suggested by Haberman (1972) can be incorporated here. We shall consider the $O(\epsilon^{\frac{1}{2}})$ jump in vorticity (as given above), and examine the conditions under which this discontinuity can be removed. The solution for Ψ_2 can be written as

$$\Psi_2 = -\tfrac{1}{6} Z^3 F_c'' + \alpha_2 Z + \int_{\pm Z_0}^{Z} \int_{\pm Z_0}^{Z} \omega_2(Z^2 + 2\zeta, \tau)\, \mathrm{d}Z\, \mathrm{d}Z,$$

and the vorticity is continuous on $Z = \pm Z_0$ if

$$\omega_2 = \begin{cases} \pm F_c''(Z^2 + 2\zeta)^{\frac{1}{2}} \mp F_c''(2\zeta_m)^{\frac{1}{2}}, & |Z| \geqslant Z_0, \\ 0, & |Z| \leqslant Z_0, \end{cases}$$

(cf. solution (28)). The solution for Ψ_2 is now as given in (28) with the additional term

$$\mp \tfrac{1}{2} F_c''(2\zeta_m)^{\frac{1}{2}} (Z \mp Z_0)^2, \quad |Z| \geqslant Z_0, \tag{61}$$

which must be matched to the outer solution. Following Haberman (1972) we express $F = U - c$ in the outer regions as

$$F = F_0(z) + \epsilon^{\frac{1}{2}} F_1^{\pm}(z, \tau), \quad F_1^{\pm}(z_c, \tau) = 0,$$

where $F_0(z)$ is our previous $F(z)$ (see the definitions after (3)). Thus we are imposing an $O(\epsilon^{\frac{1}{2}})$ distortion of the main stream; for unsteady flow this must depend on τ, and so a term $\epsilon^{\frac{1}{2}}F^{\pm}_{\frac{1}{2}\tau}$ will appear in the governing equations. The outer expansion must now take the form

$$\psi \sim \psi_0^{\pm} + \epsilon^{\frac{1}{2}}\psi_{\frac{1}{2}}^{\pm} + \epsilon\psi_1^{\pm},$$

and it then follows that $\psi_{\frac{1}{2}}^{\pm}$ satisfies

$$F_0^2\left(\frac{\psi_{\frac{1}{2}}^{\pm}}{F_0}\right)_{\xi z} + (F_{\frac{1}{2}}^{\pm})^2\left(\frac{\psi_0}{F_{\frac{1}{2}}^{\pm}}\right)_{\xi z} + F^{\pm}_{\frac{1}{2}\tau} = 0.$$

It is easy to confirm the asymptotic behaviour

$$\psi_{\frac{1}{2}} \sim A^{\pm}(\xi,\tau) + \mathscr{C}^{\pm}(\tau) \quad \text{as } z \to z_c,$$

where $\mathscr{C}^{\pm}$ is the term already introduced in (33), and

$$F^{\pm}_{\frac{1}{2}z}(z_c,\tau)\,\eta_{0\xi} + (F_c')^2 A_{\xi}^{\pm} = 0. \tag{62}$$

The new terms available for matching to (61) are therefore

$$\tfrac{1}{2}F^{\pm}_{\frac{1}{2}zc}\,Z^2 + A^{\pm} + \mathscr{C}^{\pm},$$

and so $\quad F^{\pm}_{\frac{1}{2}zc} = \mp F_c''(2\zeta_m)^{\frac{1}{2}}, \quad A^{\pm} = \mp\tfrac{1}{2}F_c''(2\zeta_m)^{\frac{1}{2}}Z_0^2, \quad \mathscr{C}^{\pm} = \pm\tfrac{1}{3}(2\zeta_m)^{\frac{3}{2}},$

with the α_2 given in (31) replaced by $\alpha_2 + F_c''(2\zeta_m)^{\frac{1}{2}}Z_0$. These choices immediately satisfy equation (62). Thus, if a distortion of the mainstream is allowed, then it is indeed possible to remove the discontinuity in vorticity. It is to be expected that this principle can be extended to the removal of higher-order discontinuities, but it would seem that no such manoeuvre will avoid the existence of the vortex sheet which is present for unsteady flows.

We now turn to one of the aims in this work, namely the description of the evolution of cats'-eyes. The surface wave was chosen to be a solution of the KdV equation, and in particular we can examine the simplest case for which a single-peaked initial profile develops into just two solitons. When the two peaks first appear (which occurs after a finite time) then the critical layer under the wave will develop a single cat's-eye. In general the n-soliton solution which evolves from a single-peaked profile will produce $n-1$ unequal cats'-eyes, but the appearance of each one will (locally) follow the pattern we shall describe here.

The evolutionary process is most clearly depicted by presenting the dominant form of the streamlines,

$$Z^2 + 2\zeta = \text{constant},$$

where $\zeta = \eta_0/F_c'^2$ and η_0 is a solution of the KdV equation (19). For the purposes of numerical computation we shall use suitably normalized variables such that

$$\eta_{0\tau} + 6\eta_0\eta_{0\xi} + \eta_{0\xi\xi\xi} = 0,$$

where $\tau \to \alpha\tau$, $\xi \to \beta\xi$ for appropriate constants α, β. The 2-soliton solution can then be written as

$$H = \frac{2\eta_0}{k_2^2} = \frac{(1+\lambda)^2[(c_+ + \mu c_-)(c_+ + c_-/\mu) - (s_+ - s_-)^2]}{(c_+ + \mu c_-)^2},$$

where $$\begin{pmatrix} c \\ s \end{pmatrix}_{\pm} = \begin{pmatrix} \cosh \\ \sinh \end{pmatrix}[(\lambda \pm 1)X - (\lambda^3 \pm 1)T],$$

with $X = \frac{1}{2}k_2\xi$, $T = \frac{1}{2}k_2^3\tau$, $\mu = (\lambda+1)/(\lambda-1)$, $\lambda = k_1/k_2$, and k_1, k_2 are the free parameters associated with the two solitons. It is now easily verified that, at $T = 0$,

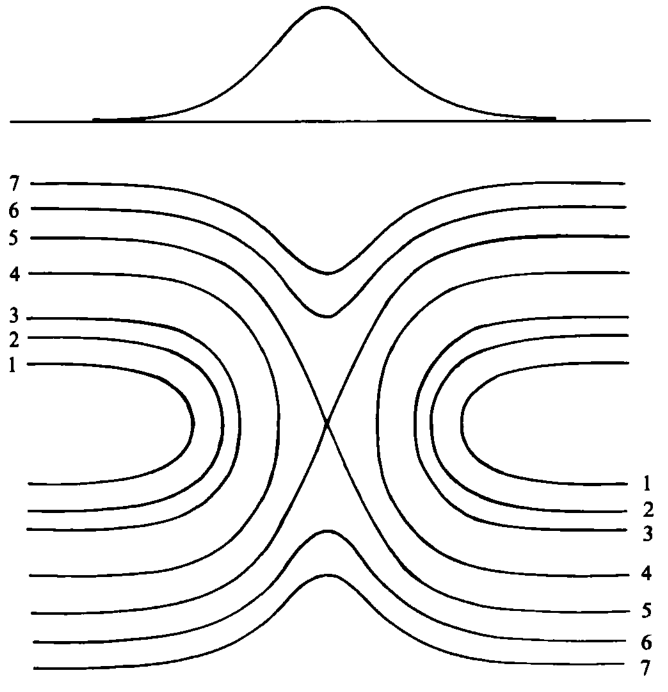

FIGURE 3. The surface wave and critical layer at $T = 0$.

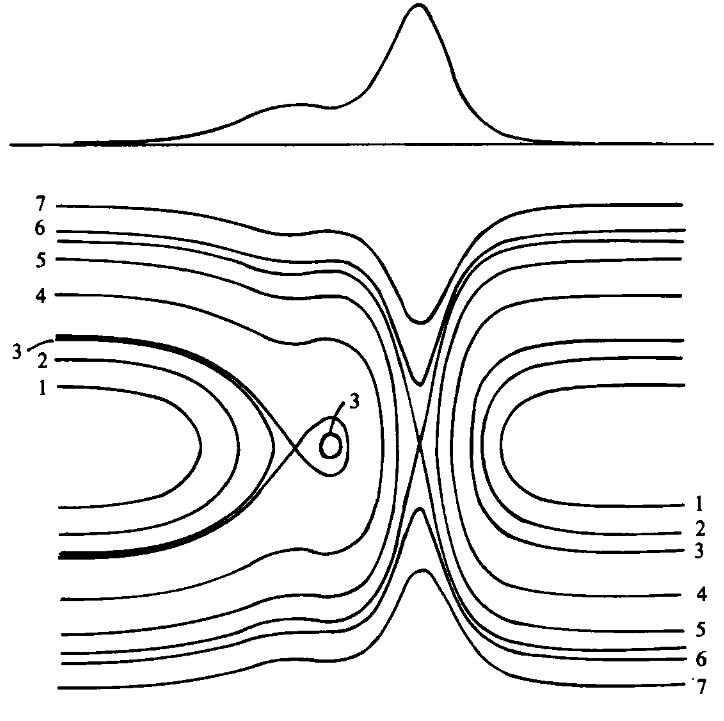

FIGURE 4. The surface wave and critical layer at $T = 0.3$; the cat's-eye has just formed.

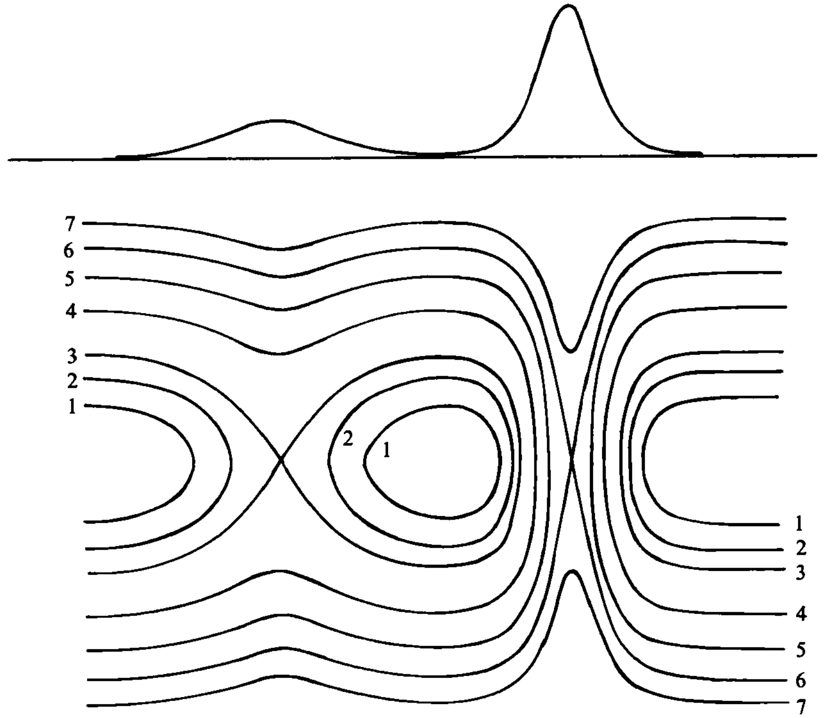

FIGURE 5. The surface wave and critical layer at $T = 1$; the two-soliton solution is for waves of amplitude 1 and 4 units.

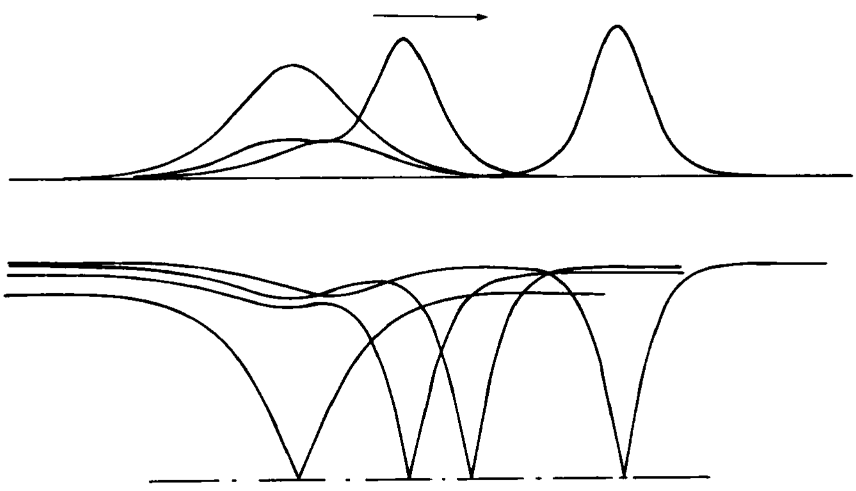

FIGURE 6. The separating streamlines (with the line of symmetry —·—) at times $T = 0, 0.3, 0.5, 1$. The surface wave is given for times $T = 0, 0.3, 1$, propagating in the direction →.

H is a symmetric profile with twin peaks if $\sqrt{3} > \lambda > 1$ but with a single peak if $\lambda \geqslant \sqrt{3}$, and the amplitudes of the two solitons are 1 and λ^2. We shall present plots of the normalized streamlines

$$Z^2 + 2H = \text{constant}$$

against X, at given T, together with the corresponding surface profile H, for $\lambda = 2$.

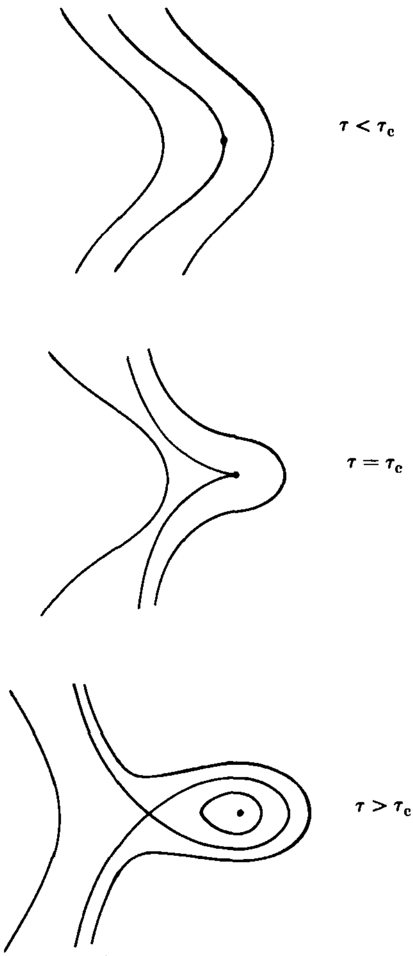

FIGURE 7. *A schematic representation of the formation of the cat's-eye for times* $\tau < \tau_c$, $\tau = \tau_c$, $\tau > \tau_c$. *The point* $\xi = \xi_c$ *is denoted by* ●.

The sequence of figures 3, 4, 5 shows the development of the 2-soliton solution from an initial single-peaked profile at $T = 0$ to the well-defined 2-soliton structure at $T = 1$. The streamline pattern clearly demonstrates the evolution of the asymmetry as T increases and in particular at $T = 0.3$ the cat's-eye has just appeared. The streamlines are numbered to indicate the constancy of stream function on the same streamline at different times. (At $T = 0.3$ additional streamlines are included to depict more completely the pattern at this time.) Since the separating streamline plays an important role in our theory, and it also evolves in time, this is presented separately in figure 6 (for times $T = 0, 0.3, 0.5, 1$) together with the surface wave. Apart from showing how the cat's-eye first appears (which we shall investigate shortly), the general picture requires no further comment except to observe that, as $T \to \infty$, so the cat's-eye elongates indefinitely. Eventually we shall have (locally) streamlines, similar to those shown in figure 4, in the neighbourhood of each peak (if our solution does persist for times greater than $O(\epsilon^{\frac{3}{2}})$).

Finally we can use our results to describe two facets of the evolution of the cat's-eye. The first follows immediately from our discussion of the vorticity and is a fundamental result here: the vorticity inside the separating streamlines is completely determined irrespective of whether this region incorporates open or closed streamlines. Hence the presence of cats'-eyes does not imply an arbitrariness in the vorticity, even for inviscid flows. The second aspect relates to the birth of a cat's-eye: using the equation for the streamlines, and the KdV equation for the surface wave, we can examine the behaviour near any point where $\eta_{0\xi} = \eta_{0\xi\xi} = 0$. Let this be at $\tau = \tau_c$, $\xi = \xi_c$; then near to this point the streamlines take the form

$$Z^2 + a(\xi - \xi_c)^3 - (\tau - \tau_c)[b + c(\xi - \xi_c)] = k,$$

approximately, where we have used the KdV equation to give η_τ and $\eta_{\xi\tau}$ at this critical point; a, b, c are positive constants, and k is a constant to be chosen. If $\tau \leqslant \tau_c$ then the streamlines cross $Z = 0$ just once, and for $\tau = \tau_c$ with $k = 0$ the streamline is a semi-cubic parabola. However, for $\tau > \tau_c$ some streamlines cross $Z = 0$ twice and for one special value of k the curve is a strophoid: this is the boundary of the cat's-eye. The sequence is sketched in figure 7, where the transition to a strophoid constitutes the birth process. Although other models, besides the KdV equation, are no doubt possible, the local behaviour with $\eta_{0\xi} = \eta_{0\xi\xi} = 0$ but $\eta_{0\xi\xi\xi} \neq 0$ would seem a satisfactory requirement for the generation of a cat's-eye. It is, of course, outside the scope of this study to attempt any examination of the generic problem for the formation of cats'-eyes.

The author is grateful to two referees who highlighted some shortcomings in an earlier draft of this paper.

Appendix

The expansion of $\psi_0^\pm + \epsilon\psi_1^\pm$ for $z = z_c + \epsilon^{\frac{1}{2}} Z$, as $\epsilon \to 0^+$, is

$$\begin{aligned}
\psi_0^\pm + \epsilon\psi_1^\pm = {} & \frac{1}{F'_c}\,\eta_0^\pm + \epsilon^{\frac{1}{2}} \ln\epsilon \left(\frac{1}{2}\frac{F''_c}{F'^2_c} Z\eta_0^\pm\right) + \epsilon^{\frac{1}{2}} \left[Z \left(\frac{F''_c}{F'^2_c} \ln|Z| + \frac{1}{2}\frac{F''_c}{F'^2_c} - F'_c \not{I}^\pm \right)\eta_0^\pm \right. \\
& \left. - \frac{1}{4}\frac{F''_c}{F'^4_c}\frac{1}{Z}(\eta_0^\pm)^2 \right] + \epsilon(\ln\epsilon)^2 \left(\frac{1}{8}\frac{F''^2_c}{F'^5_c}(\eta_0^\pm)^2 \right) + \epsilon\ln\epsilon \left[\frac{1}{4}\frac{F''^2_c}{F'^3_c} Z^2 \eta_0^\pm \right. \\
& \left. + \frac{1}{2}\left(\frac{F''^2_c}{F'^5_c}\ln|Z| + \frac{3}{2}\frac{F''^2_c}{F'^5_c} - \frac{F''_c}{F'^2_c}\not{I}^\pm \right)(\eta_0^\pm)^2 - \frac{1}{2}\frac{F''_c}{F'^3_c}\int_\xi^\infty \eta_{0_\tau}^\pm \,\mathrm{d}\xi \right] \\
& + \epsilon \left[\frac{1}{F'_c}\,\eta_1^\pm + \tfrac{1}{2}Z^2 \left(\frac{F''^2_c}{F'^3_c}\ln|Z| + \frac{F'''_c}{F'^2_c} - \frac{3}{2}\frac{F''^2_c}{F'^3_c} - F''_c \not{I}^\pm \right)\eta_0^\pm \right. \\
& - \left(\frac{F''_c}{F'^3_c}\ln|Z| + \frac{5}{2}\frac{F''_c}{F'^3_c} - \not{I}^\pm \right) \int_\xi^\infty \eta_{0_\tau}^\pm \,\mathrm{d}\xi \pm \frac{1}{F'_c} K^\pm \eta_{0\xi\xi}^\pm + \frac{1}{2}\left\{ F'_c \left(\frac{F''_c}{F'^3_c}\ln|Z| - \not{I}^\pm \right)^2 \right. \\
& \left.\left. + 3\frac{F''_c}{F'^2_c}\left(\frac{F''_c}{F'^3_c}\ln|Z| - \not{I}^\pm \right) + \frac{11}{2}\frac{F''^2_c}{F'^5_c} - \frac{F'''_c}{F'^4_c} \right\} (\eta_0^\pm)^2 \right] + \epsilon^{\frac{3}{2}}(\ln\epsilon)^2 \left[\frac{1}{8}\frac{F''^3_c}{F'^6_c} Z(\eta_0^\pm)^2 \right] \\
& + \epsilon^{\frac{3}{2}}\ln\epsilon \left[\frac{1}{12}\frac{F''_c F'''_c}{F'^3_c} Z^3\eta_0^\pm + \frac{1}{2}\frac{F''_c}{F'^2_c} Z\eta_1^\pm - \frac{1}{2}\left(3\frac{F''^2_c}{F'^4_c} - \frac{F'''_c}{F'^3_c} \right) Z \int_\xi^\infty \eta_{0_\tau}^\pm \,\mathrm{d}\xi \right. \\
& \left. \pm \frac{1}{2}\frac{F''_c}{F'^2_c} ZK^\pm \eta_{0\xi\xi}^\pm + \tfrac{1}{2}Z \left(\frac{F''^3_c}{F'^6_c}\ln|Z| + \frac{1}{4}\frac{F^{\mathrm{iv}}_c}{F'^4_c} + \frac{11}{4}\frac{F''^3_c}{F'^6_c} - \frac{3}{2}\frac{F''_c F'''_c}{F'^5_c} - \frac{F''^2_c}{F'^3_c}\not{I}^\pm \right)(\eta_0^\pm)^2 \right] \\
& + \epsilon^{\frac{3}{2}} \left[Z \left(\frac{F''_c}{F'^2_c}\ln|Z| + \frac{1}{2}\frac{F''_c}{F'^2_c} - F'_c \not{I}^\pm \right) \eta_1^\pm \right.
\end{aligned}$$

$$
\begin{aligned}
&+\tfrac{1}{6}Z^3\left(\frac{F''_c F'''_c}{F'^3_c}\ln|Z|+\frac{1}{2}\frac{F^{\mathrm{iv}}_c}{F'^2_c}-\frac{1}{2}\frac{F''_c F'''_c}{F'^3_c}-\frac{3}{4}\frac{F''^3_c}{F'^4_c}-F'''_c \not{I}^{\pm}\right)\eta_0^{\pm} \\
&-Z\left\{\left(3\frac{F''^2_c}{F'^4_c}-\frac{F'''_c}{F'^3_c}\right)\ln|Z|+\frac{1}{3}\frac{F'''_c}{F'^3_c}-\frac{3}{4}\frac{F''^2_c}{F'^4_c}\mp 2F'_c I_3^{\pm}\right\}\int_{\xi}^{\infty}\eta_{0_\tau}^{\pm}\,\mathrm{d}\xi \\
&+Z\left\{\pm\frac{F''_c}{F'^2_c}K^{\pm}\ln|Z|+F'_c\left(K^{\pm}I_2^{\pm}-L^{\pm}\pm\frac{1}{2}\frac{F''_c}{F'^3_c}K^{\pm}\right)\right\}\eta_{0\xi\xi}^{\pm} \\
&+Z\left\{\tfrac{1}{2}F''_c\left(\frac{F''_c}{F'^3_c}\ln|Z|-\not{I}^{\pm}\right)^2+F'_c\left(\frac{F''^2_c}{F'^4_c}-\frac{F'''_c}{F'^3_c}\right)\left(\not{I}^{\pm}-\frac{F''_c}{F'^3_c}\ln|Z|\right)\right. \\
&-\frac{1}{4}\left(10\frac{F''_c F'''_c}{F'^5_c}-15\frac{F''^3_c}{F'^6_c}-\frac{F^{\mathrm{iv}}_c}{F'^4_c}\right)\ln|Z|+\frac{1}{8}\frac{F^{\mathrm{iv}}_c}{F'^4_c}+\frac{1}{3}\frac{F''_c F'''_c}{F'^5_c}-\frac{F''^3_c}{F'^6_c} \\
&\left.\pm\tfrac{3}{2}F'_c I_4^{\pm}\right\}(\eta_0^{\pm})^2\Bigg]+o(\epsilon^{\frac{3}{2}}).
\end{aligned}
$$

The notation used here is given in the paper: in particular see §§2, 3.

REFERENCES

Batchelor, G. K. 1956 *J. Fluid Mech.* **1**, 177–190.

Benjamin, T. B. 1962 *J. Fluid Mech.* **12**, 97–116.

Benney, D. J. & Bergeron, R. F. 1969 *Stud. Appl. Maths* **48**, 181–204.

Benney, D. J. & Maslowe, S. A. 1975 *Stud. Appl. Maths* **54**, 181–205.

Brown, S. N. & Stewartson, K. 1978 *Geophys. Astrophys. Fluid Dyn.* **10**, 1–24.

Brown, S. N. & Stewartson, K. 1979 *Geophys. Astrophys. Fluid Dyn.* **14**, 1–18.

Burns, J. C. 1953 *Proc. Camb. Phil. Soc.* **49**, 695–706.

Davis, R. E. 1969 *J. Fluid Mech.* **36**, 337–346.

Freeman, N. C. & Johnson, R. S. 1970 *J. Fluid Mech.* **42**, 401–409.

Haberman, R. 1972 *Stud. Appl. Maths* **51**, 139–161.

Kelvin, Lord 1880 *Nature* **23**, 45–46.

Margolis, S. B. & Su, C. H. 1978 *Phys. Fluids* **21**, 1247–1259.

Maslowe, S. A. & Redekopp, L. G. 1980 *J. Fluid Mech.* **101**, 321–348.

Moore, D. W. & Saffman, P. G. 1982 *Proc. R. Soc. Lond.* A **382**, 389–410.

Redekopp, L. G. 1977 *J. Fluid Mech.* **82**, 725–745.

Reid, W. H. 1965 *The Stability of Parallel Flows. Basic Developments in Fluid Dynamics*, vol. 1, pp. 249–308. Academic.

Stewartson, K. 1978 *Geophys. Astrophys. Fluid Dyn.* **9**, 185–200.

Varley, E. & Blythe, P. A. 1983 *Stud. Appl. Maths* **68**, 103–188.

Velthuizen, H. G. M. & van Wijngaarden, L. 1969 *J. Fluid Mech.* **39**, 817–829.